The Interbank market and the core-periphery structure in networks

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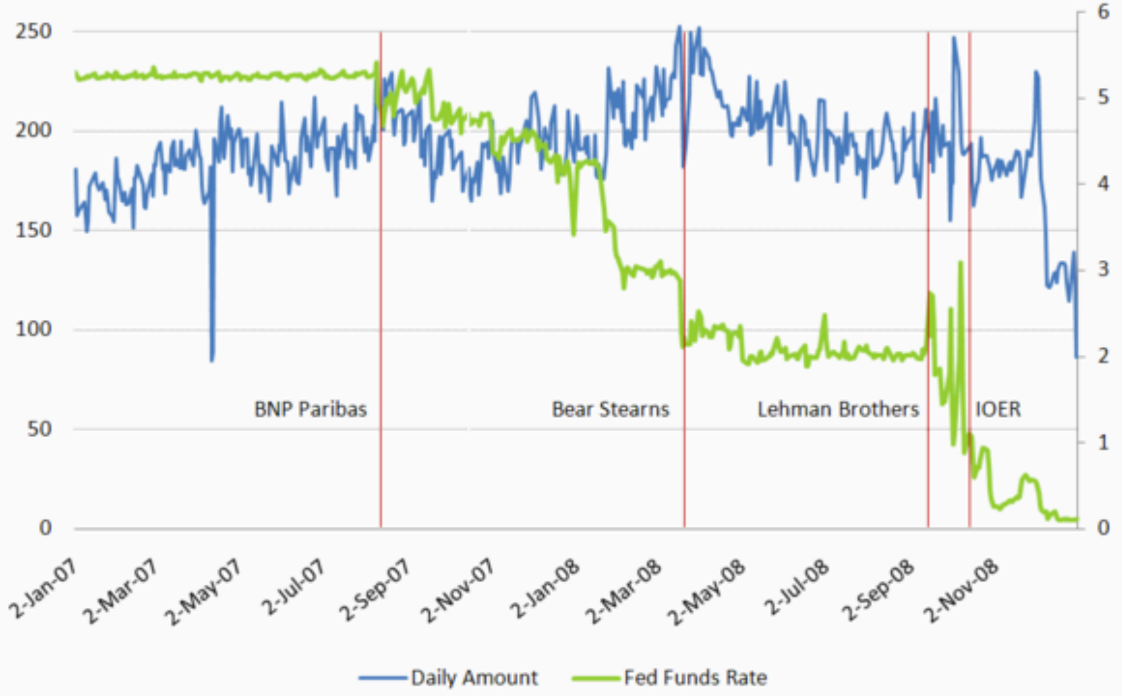
During the onset of the subprime financial crisis of 2007-2008 a key role was played by the Interbank market.

- August 9th, 2007 – BNP Paribas limits withdrawals,
- March 16th, 2008 – JP Morgan announces Bear Stearns acquisition,
- September 15th, 2008 – Lehman Brothers files for bankruptcy, and
- October 9th, 2008 – First effective day of interest on reserve balances.

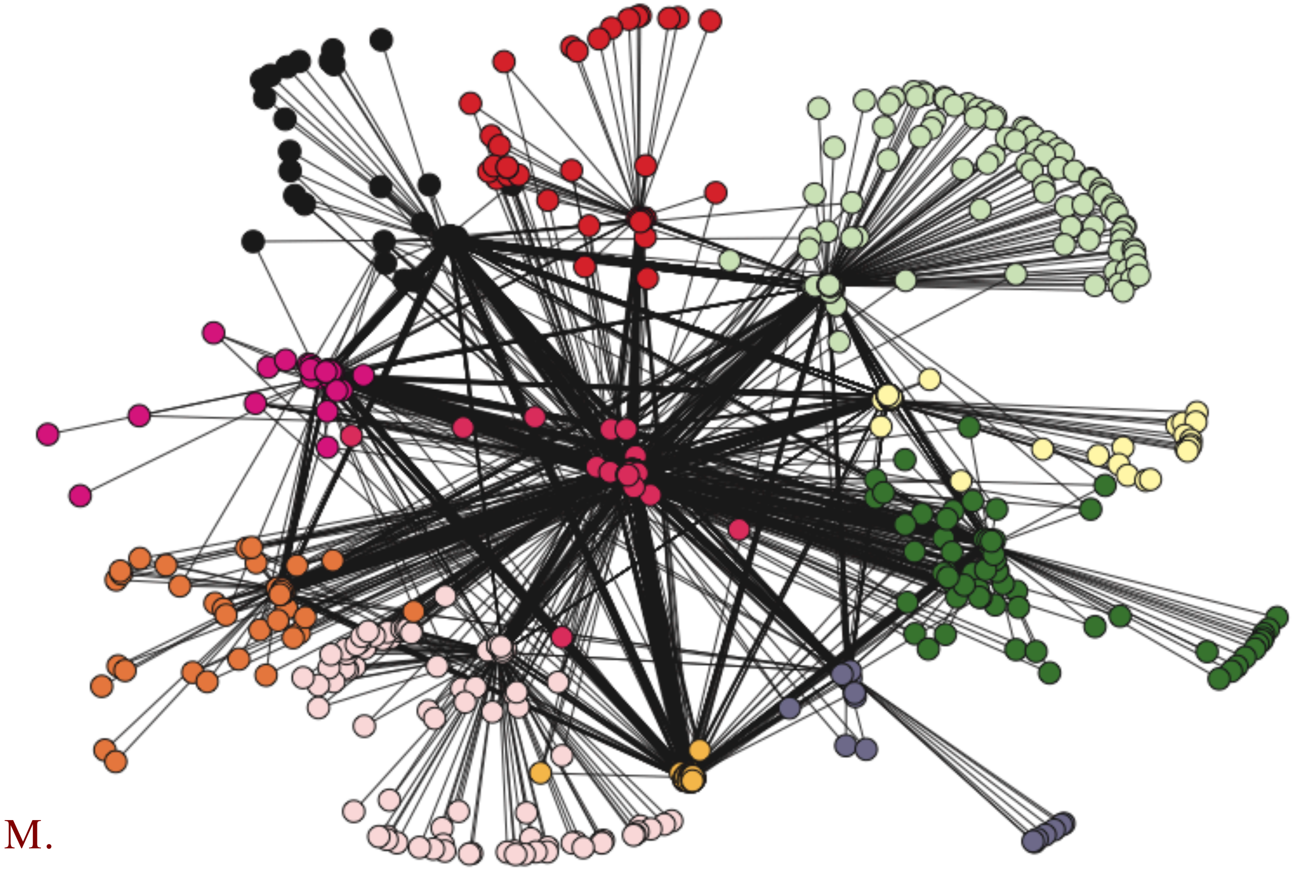
Figure 1. Daily amount ($ billions) and daily fed funds rate



Afonso, G., Kovner, A. and Schoar, A., 2010. What happened to US interbank lending in the financial crisis? Vox.

Afonso, G., Kovner, A. and Schoar, A., 2011. Stressed, not frozen: The federal funds market in the financial crisis. The Journal of Finance, 66(4), pp.1109-1139.

The interbank market has been investigated with tools of network science since 2004



(a)

FIG. 1: The banking network of Austria (a). Clusters are grouped (colored) according to regional and sectorial organization: R-sector with its federal state sub-structure: RB yellow, RSt orange, light orange RK, gray RV, dark green RT, black RN, light green RO, light yellow RS. VB-sector: dark gray, S-sector: orange-brown, other: pink. Data is from the September 2002 L matrix, which is representative for all

Boss, M., Elsinger, H., Summer, M. and Thurner, S., 2004. Network topology of the interbank market. Quantitative Finance, 4(6), pp.677-684.

After the subprime crisis a series of studies have investigated the systemic risk associated with the interlinkages observed in the interbank market.

One of the most used model is the Eisenberg-Noe model

In this model banks are nodes and the links between pair of nodes are defined by a *n* x *n* liabilities matrix P where $p_{ij} \geq 0$ represents the payment due from bank i to bank j.

In addition to the interbank obligations each node holds outside (to the network) assets c_i and outside liabilities b_i

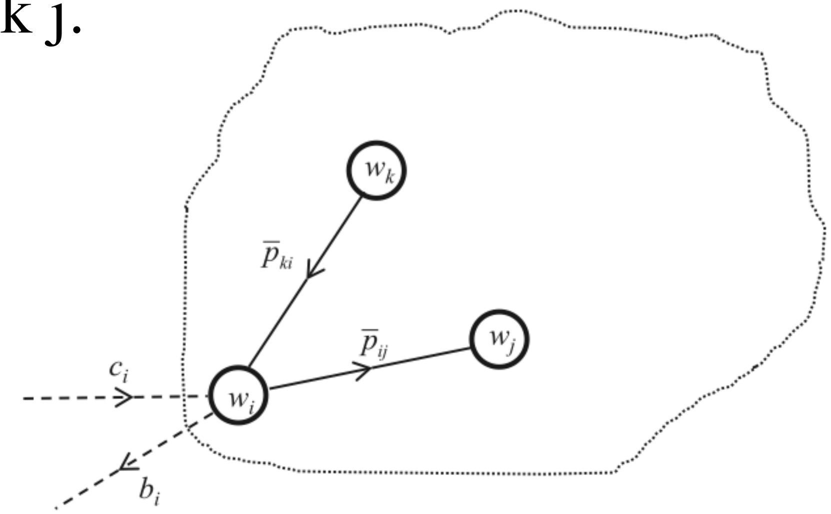


Fig. 1. Node *i* has an obligation $\bar{p}_{ij}$ to node *j*, a claim $\bar{p}_{ki}$ on node *k*, outside assets c_i, and outside liabilities b_i, for a net worth of w_i.

The net worth of each bank is

$$w_i = \overset{\text{outside assets}}{c_i} + \underset{\text{interbank assets}}{\sum_{j\neq i} p_{ji}} + \overset{\text{outside liabilities}}{b_i} + \underset{\text{interbank liabilities}}{\sum_{j\neq i} p_{ij}}$$

When the net worth goes at or below zero the bank goes bankrupt and the connected (and also disconnected) banks feel an impact through two basic distinct channels.

Eisenberg, L. and Noe, T.H., 2001. Systemic risk in financial systems. Management Science, 47(2), pp.236-249.

Glasserman, P. and Young, H.P., 2015. How likely is contagion in financial networks?. Journal of Banking & Finance, 50, pp.383-399.

1) spillover or "domino" effect. This is just the spreading of the financial distress due to the losses felt by other banks due to the impossibility of the failed bank to fulfill its obligations.

2) fire sales. The triggering of a fire sale can change the mark-to market value of the portfolio of assets held by the distressed bank and this can change the asset value of all other banks having portfolios overlapping with the one of the distressed bank.

Most of the studies have considered only one of these channels whereas typically both are present in reality.

The first study about domino effect

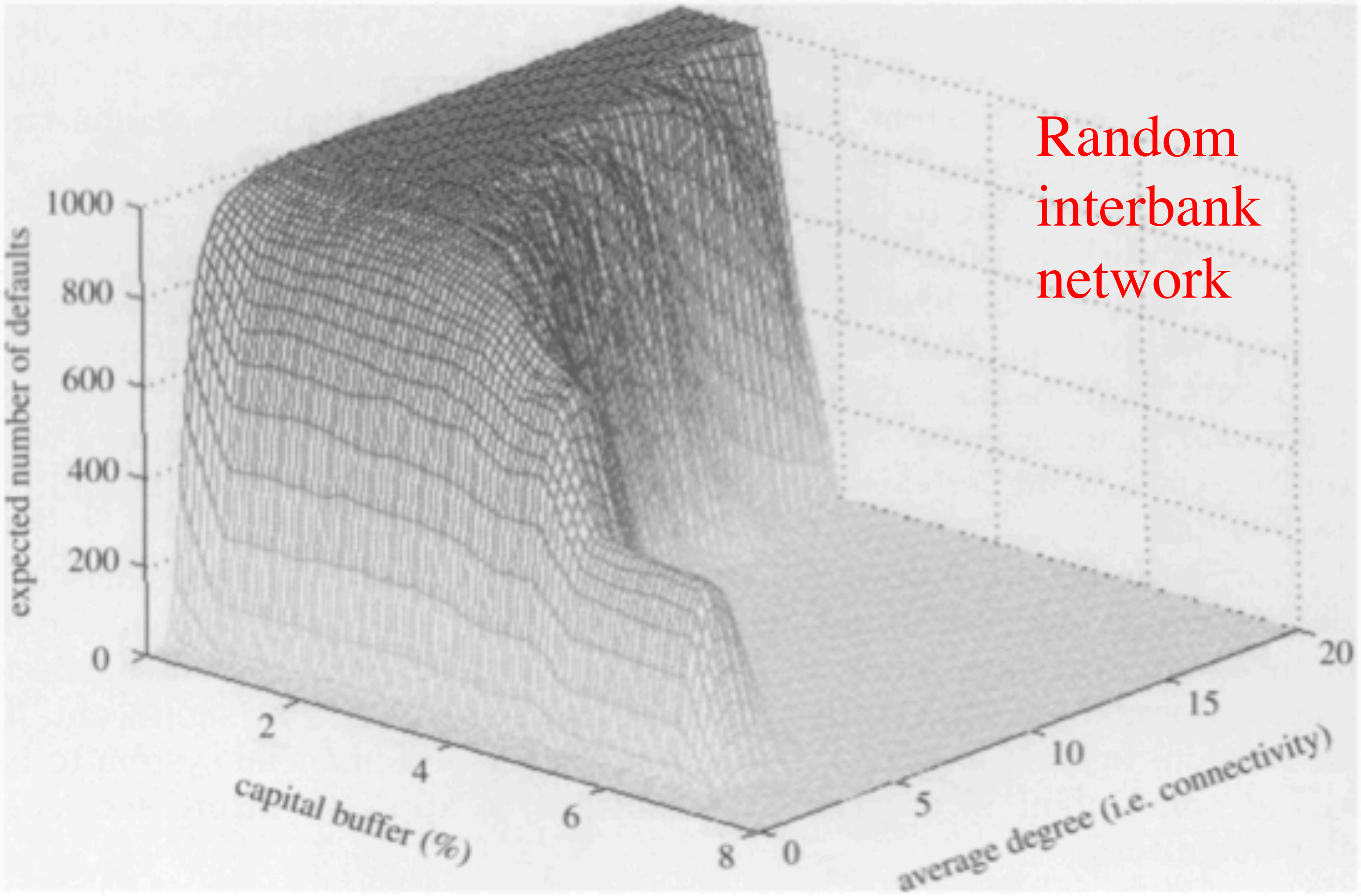


Figure 6. Connectivity, capital buffers and the expected number of defaults.

Gai, P. and Kapadia, S., 2010, March. Contagion in financial networks. In Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences (p. rspa20090410). The Royal Society.

Empirical characterization of the interbank market

- Different network models considered;

- Problems related to the empirical characterization;

- Trust building and time evolving trust relation in the interbank market.

The topology of the interbank market has been described as compatible with two classes of network models:

- scale free networks;

- core-periphery networks.

The concept of core-periphery network structure originates in social networks. Examples are:

- Study of national elites and collective action (Laumann and Pappi, 1976; Alba and Moore, 1978),
- interlocking directorates (Mintz and Schwartz, 1981),
- scientific citation networks (Mullins et al., 1977; Doreian, 1985),
- proximity among Japanese monkeys (Corradino, 1990).

- Laumann, E.O., Pappi, F.U., 1976. Networks of Collective Action: A Perspective on Community Influence Systems. Academic Press, New York.
- Alba, R.D., Moore, G., 1978. Elite social circles. Sociological Methods and Research 7, 167–188.
- Mintz, B., Schwartz, M., 1981. Interlocking directorates and interest group formation. American Sociological Review 46, 851–868.
- Mullins, N.C., Hargens, L.L., Hecht, P.K., Kick, E.L., 1977. The group structure of cocitation clusters: a comparative study. American Sociological Review 42, 552–562.
- Doreian, P., 1985. Structural equivalence in a psychology journal network. American Society for Information Science 36, 411–417.
- Corradino, C., 1990. Proximity structure in a captive colony of Japanese monkeys (Macaca fuscata fuscata) an application of multidimensional scaling. Primates 31, 351–362.

The first formalization of the core-periphery structure was done by Borgatti and Everett[¶]

S.P. Borgatti, M.G. Everett / Social Networks 21 (1999) 375–395

In a core-periphery network one observes a core of highly connected nodes and a set of nodes that are connected to some or all element of the core and poorly connected among them.

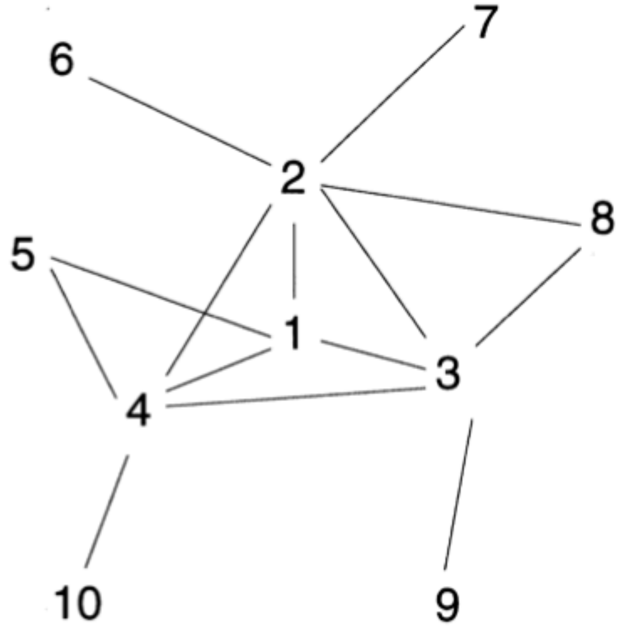
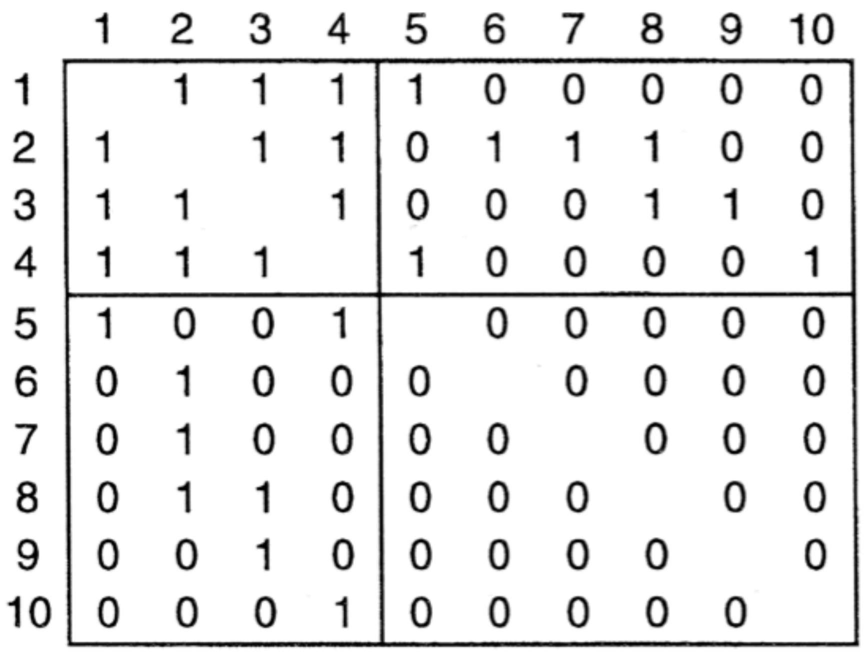


Fig. 1. A network with a core/periphery structure.

[¶] Borgatti, S. P., & Everett, M. G. (2000). Models of core/periphery structures. Social networks, 21(4), 375-395.

In their original paper Borgatti and Everett use the adjacency matrix to explain the concept.

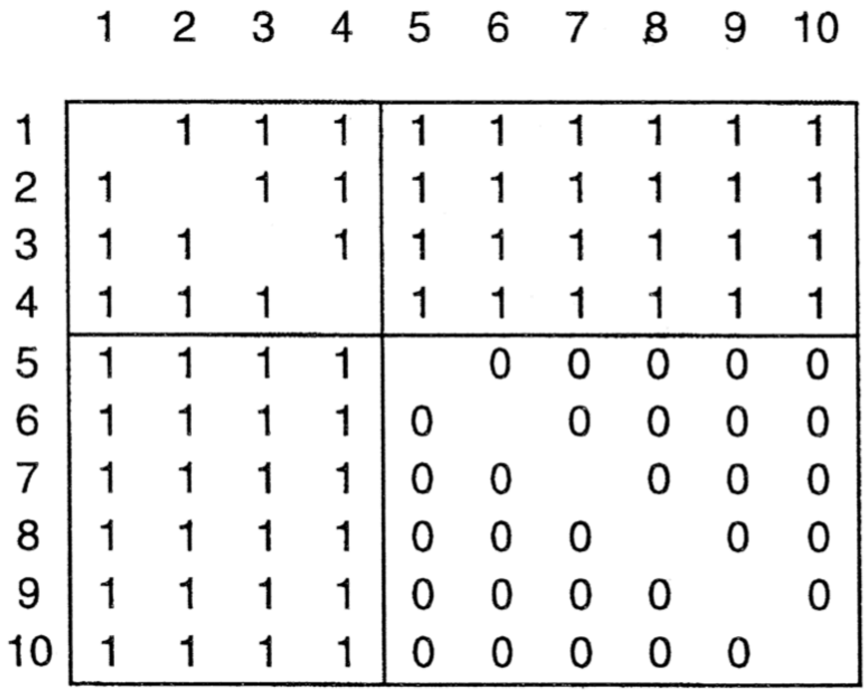
The previous example of core-periphery network



	1	2	3	4	5	6	7	8	9	10
1		1	1	1	1	0	0	0	0	0
2	1		1	1	0	1	1	1	0	0
3	1	1		1	0	0	0	1	1	0
4	1	1	1		1	0	0	0	0	1
5	1	0	0	1		0	0	0	0	0
6	0	1	0	0	0		0	0	0	0
7	0	1	0	0	0	0		0	0	0
8	0	1	1	0	0	0	0		0	0
9	0	0	1	0	0	0	0	0		0
10	0	0	0	1	0	0	0	0	0	

Table 1
The adjacency matrix of Fig. 1

Idealized core-periphery network



	1	2	3	4	5	6	7	8	9	10
1		1	1	1	1	1	1	1	1	1
2	1		1	1	1	1	1	1	1	1
3	1	1		1	1	1	1	1	1	1
4	1	1	1		1	1	1	1	1	1
5	1	1	1	1		0	0	0	0	0
6	1	1	1	1	0		0	0	0	0
7	1	1	1	1	0	0		0	0	0
8	1	1	1	1	0	0	0		0	0
9	1	1	1	1	0	0	0	0		0
10	1	1	1	1	0	0	0	0	0	

Table 2
Idealized core/periphery structure

Everett and Borgatti introduced a few measures of core-periphery structure. The most basic one is the following:

$$\rho = \sum_{i,j} a_{ij}\delta_{ij}$$

where **A** is the adjacency matrix of the real network and **Δ** is the is the adjacency matrix of an idealized core-periphery structure defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } \mathrm{c}_i = \text{core or } \mathrm{c}_j = \text{core} \\ 0 & \text{otherwise} \end{cases}$$

Is core-periphery structure indicating a deviation from random matching? Short answer is "not always".

In fact Chung and Lu (2002) showed that power-law random graphs with degree distribution proportional to $k^{-\beta}$ with exponent β in the interval [2,3] almost surely contain a dense subgraph (i.e. a core) that has short distance to almost all other nodes.

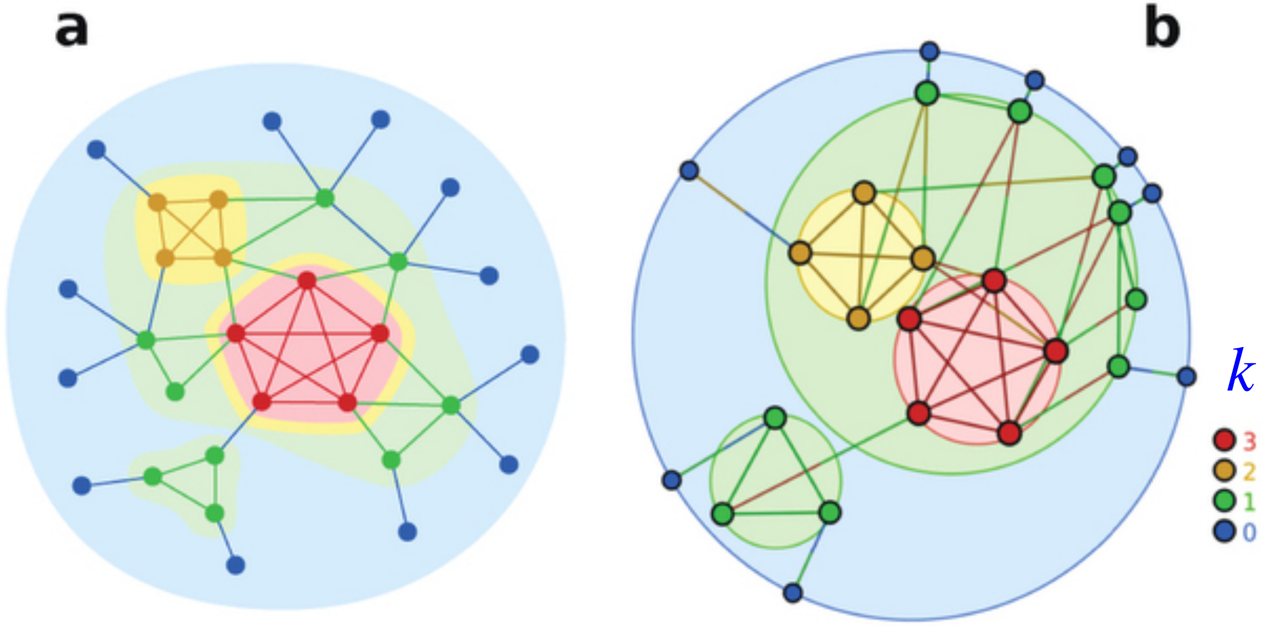
It is therefore important to qualify whether the presence of a core is related to aspects of the network settings or just consistent with a power-law null model.

Chung, Fan, and Linyuan Lu. "The average distances in random graphs with given expected degrees." PNAS 99 (2002): 15879-15882.

Petter Holme (2005) proposed a more general core-periphery coefficient taking into account a comparison with a corresponding configuration model

The new definition consider as the basic cluster (core) a *k*-core (Holme describes it as "the most rudimentary cluster definition").

A *k*-core of a connected component is the subnetwork obtained after all vertices of degree less than *k* have been removed



In the search of *k*-cores a sequence of *k*-cores is obtained.

Holme, Petter. "Core-periphery organization of complex networks." Physical Review E 72 (2005): 046111.

Core-periphery structures in directed networks

P. CSERMELY *ET AL.*

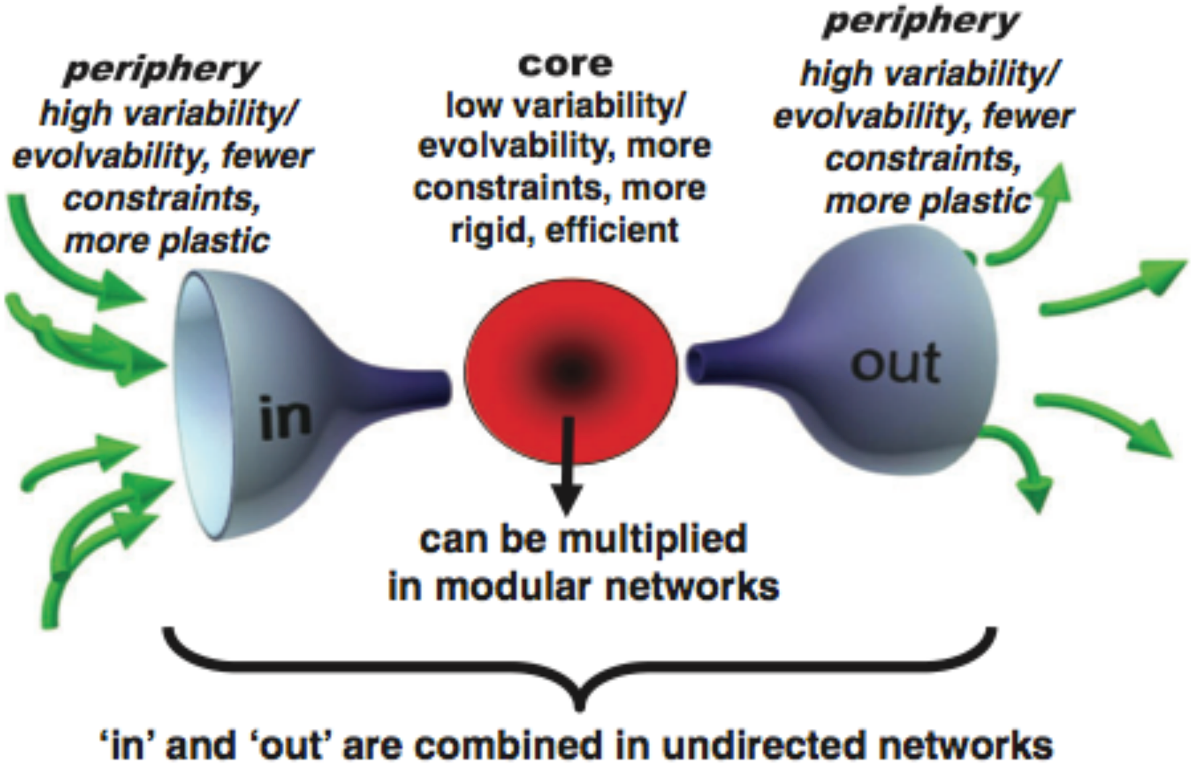
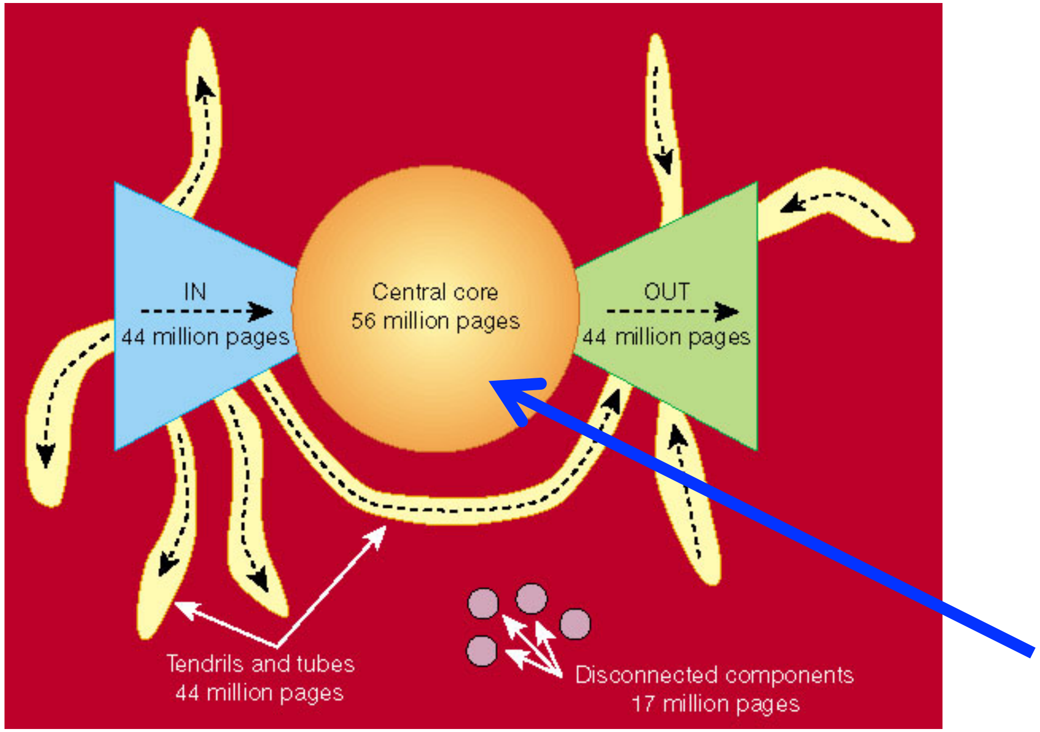


FIG. 1. General features of core/periphery network structures shown by the example of the bow-tie architecture of directed networks. The 'in' and 'out' components of network periphery refer to the fan-in and fan-out segments of bow-tie networks containing source and sink nodes, respectively. These segments of network periphery are combined in undirected networks. The network periphery has higher variability, dynamics and evolvability, has fewer constraints, and is more plastic than the core. Network cores facilitate system robustness helping the adaptation to large fluctuations of the environment, as well as to noise of intrinsic processes. The network core can be regarded as a highly degenerate segment of the complex system, where the densely intertwined pathways can substitute and/or support each other. The network core has lower variability, dynamics and evolvability, and is more rigid and more efficient than the periphery. Core structures may be multiplied in modular networks. Adapted from Tieri *et al*. [13].

By using information on over 200 million pages and 1.5 billion links of 1999 data, Broder et al provided an overview of the structure of the WWW by dividing the web into a few large pieces and by showing how the pieces fit in a stylized way.



Main finding:

- the web contains a giant strongly connected component, all major web pages can mutually reach one other;

¶Andrei Broder, Ravi Kumar, Farzin Maghoul, Prabhakar Raghavan, Sridhar Rajagopalan, Raymie Stata, Andrew Tomkins, and Janet Wiener. Graph structure in the Web. In Proc. 9th International World Wide Web Conference, pages 309-320, 2000.

A recent study focusing on the core-periphery structure[¶]

Economic motivation:
- the detection of a tiering structure in the interbank market (i.e. an organization in layers).

Working hypothesis:
The banks in the market are partitioned into two sets based on the type of their bilateral relations with each other
(i) top-tier banks lend to each other,
(ii) lower-tier banks do not lend to each other,
(iii) top-tier banks lend to lower-tier banks, and
(iv) top-tier banks borrow from lower-tier banks.

[¶] Craig, B., & Von Peter, G. (2014). Interbank tiering and money center banks. Journal of Financial Intermediation, 23, 322-347.

Craig and von Peter describe the system in terms of a blockmodel with a set of top-tier banks (the "core" (C)) and a set of lower-tier banks (the "periphery" (P)).

Formally the block model is defined in terms of the following adjacency matrix M

$$\mathbf{M} = \begin{pmatrix} CC & CP \\ PC & PP \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{RR} \\ \mathbf{CR} & \mathbf{0} \end{pmatrix}$$

where **RR** is a regular block which contains at least one link in every row. **CR** is a column regular matrix with at least one 1 in every column.

Fitting a core-periphery network model to the empirical adjacency matrix.

Craig and von Peter define the following function as error score

$$e = \frac{e_{cc} + e_{pp} + e_{cp} + e_{pc}}{\sum_i \sum_j E_{ij}}$$

where e_{cc} is an error in the core, e_{pp} in the periphery, e_{cp} (e_{pc}) in the interlinkages between core periphery (periphery core). E_{ij} is 1 when an edge is present between node i and j in the empirical network and 0 otherwise.

The optimal partition of nodes in core-periphery is the one found for the model **M** with the lowest error score.

Stylized examples of a model and of diagonal and off-diagonal block errors

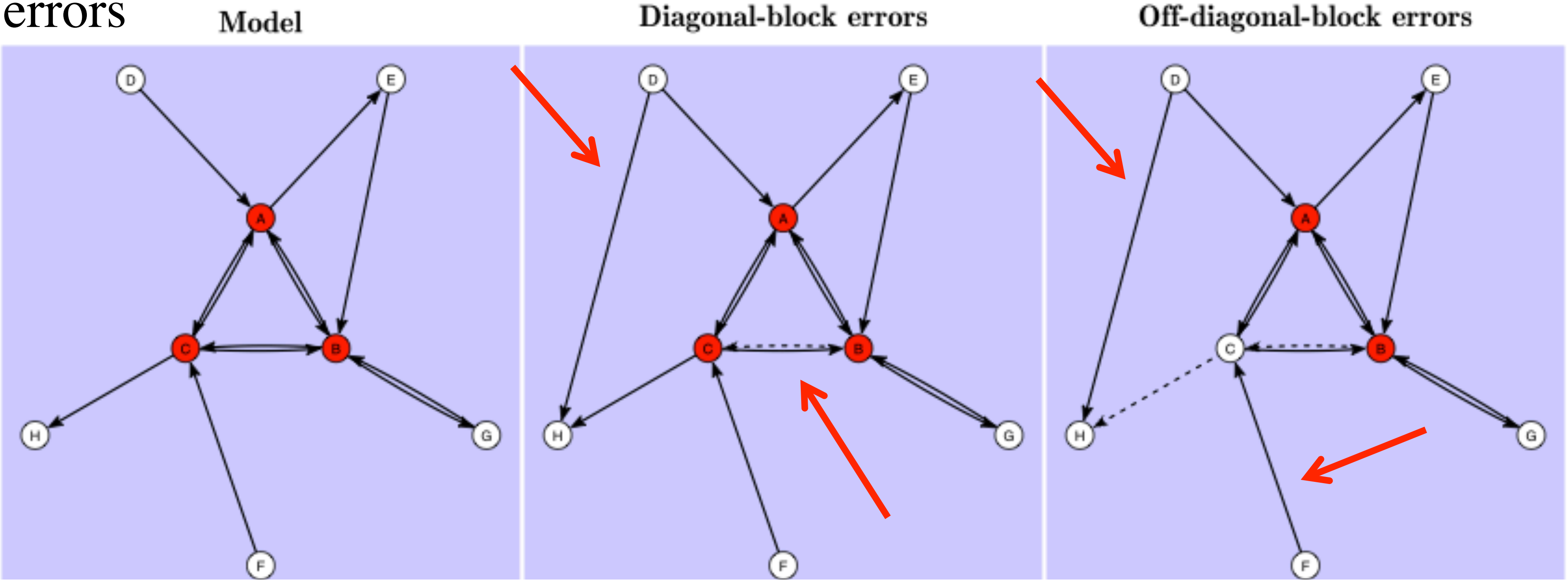


Fig. 1. Stylized example of an interbank market. The left panel illustrates a perfectly tiered interbank structure in a stylized interbank market comprising 8 banks. The arrows represent the direction of credit exposure, e.g. bank D lends to A. The middle and right panels depict examples of networks that are not perfectly tiered.

$$\begin{pmatrix} \mathbf{1} & \mathbf{RR} \\ \mathbf{CR} & \mathbf{0} \end{pmatrix} = \left(\begin{array}{ccc|ccccc} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

core {A,B,C}

$e_{cc}=1$, $e_{pp}=1$

$e=2/13$

core {A,B}

$e_{pp}=2$

$e=2/12$

Empirical investigation of the German banking system

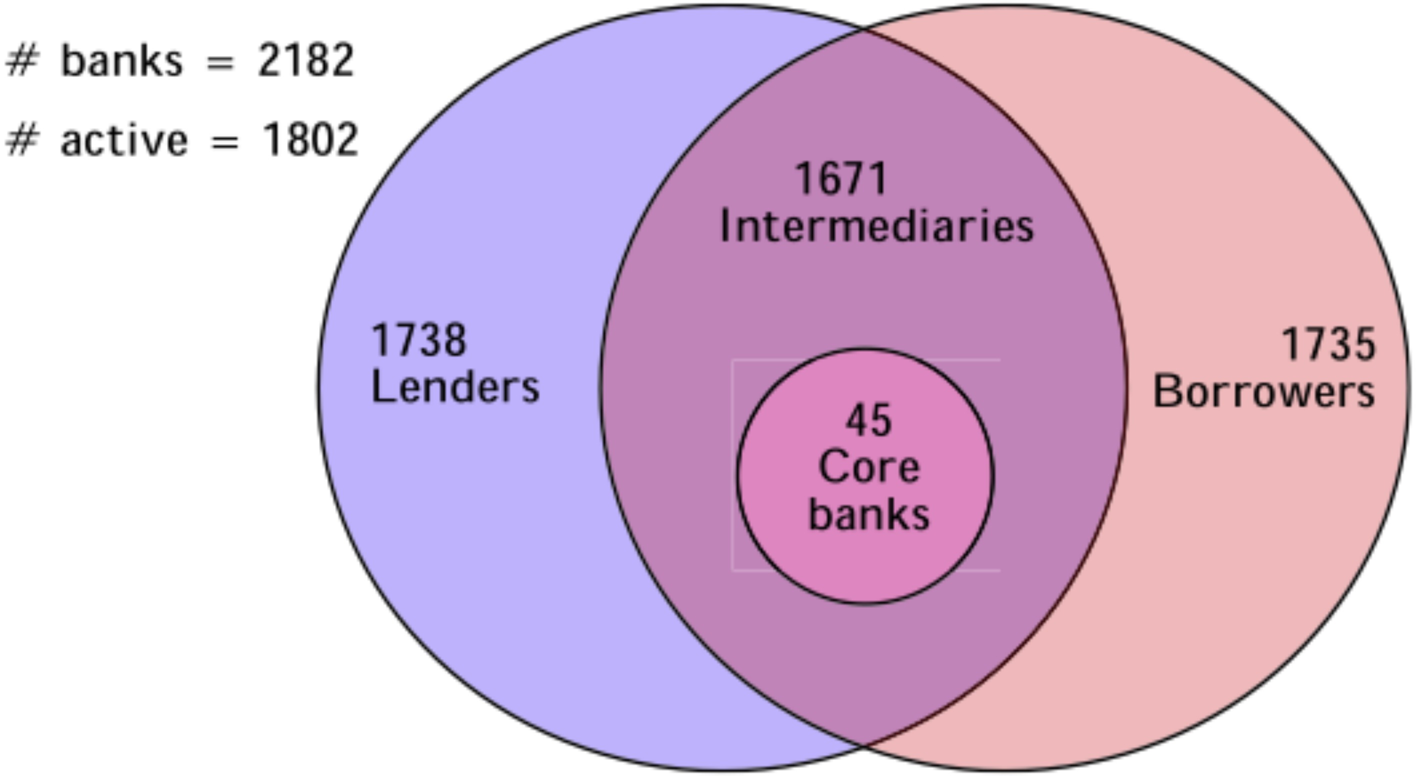
- bilateral interbank positions between more than 2000 banks,
- quarterly analysis during the time period from 1999 Q1 to 2012 Q3,
- one of the largest banking systems in the world with assets totaling EUR 7.6 trillion at the end of 2007,
- the number of active banks in the interbank market equals 1732 ± 85 (mean ± SD),
- the set of banks comprises, on average, 40 private credit banks, 400 saving banks, 1150 credit unions, and 200 special purposes banks,
- no distinction is done about the maturity of the credit relationships,
- the value of the credit relationships is not used to detect the core-periphery model but it is used to interpret it.

Empirical results about a representative pre-crisis mid-sample quarter (2003 Q2).

Out of 2182 banks, 1802 are active in the interbank market. 1671 as intermediaries (i. e. lending and borrowing), 67 as lenders only and 64 as borrowers only.

Finding the optimal model M by numerical search is a large-scale problem in combinatorial optimization. Exhaustive search cannot be performed. Craig and von Peter use a sequential optimization algorithm. It is a type of greedy algorithm starting from a random initial partition and performing optimization by switching nodes from the core to the periphery and vice versa until the error score is minimized.

Using this numerical procedure authors found fpr 2003 Q2 an optimal model with a core consisting of 45 banks. This is 2.7% of intermediaries.



The error score was 0.122 (i.e. 12.2% of the links of the networks were mismatched (2406 of 19720).

The statistical fit of the core-periphery model is robust and consistent over time

B. Craig, G. von Peter/J. Finan. Intermediation 23 (2014) 322–347

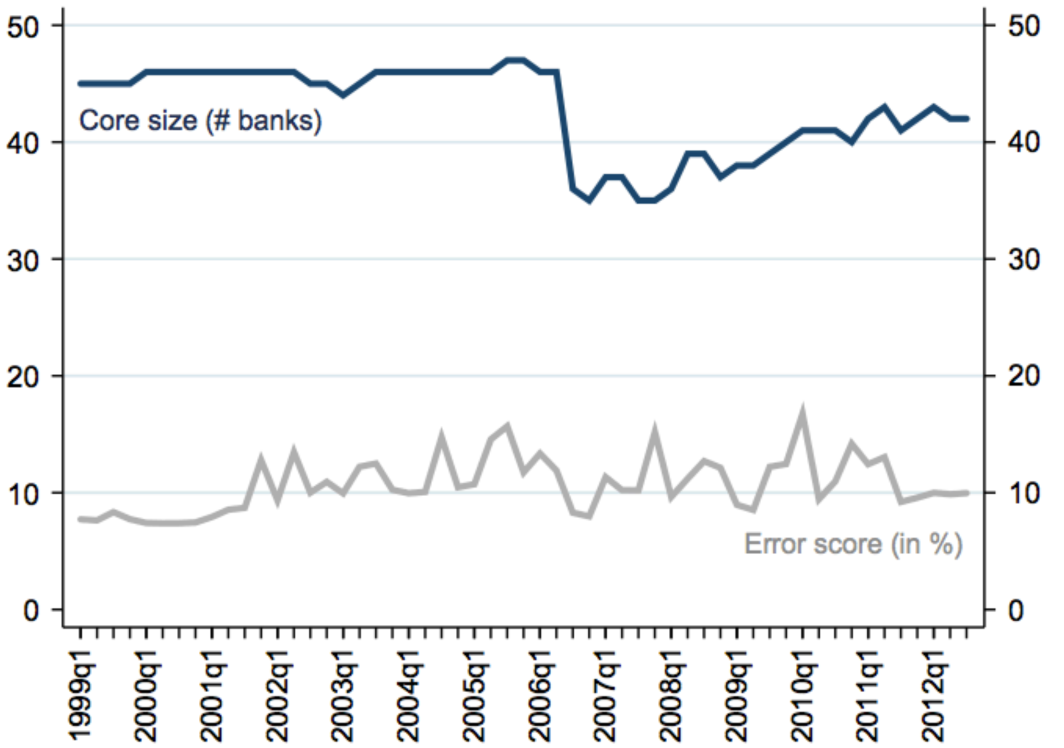


Fig. 3. Structural stability over time. The figure shows a quarterly time series of the size of the estimated core and the total error score expressed as a percentage of links as in Eq. (3), for the German interbank network.

Stability and consistency is observed also across the 2007 onset of financial crisis.

It is also worth noting that the composition of banks within the core (and therefore in the periphery) remain stable across quarters.

In fact, the conditional probability of observing a given bank in a set s also in the set s' of another quarter is equal to

$$P\left(s'|s\right)=\left(\begin{array}{cccc} & core & per. & out \\ core & 0.931 & 0.060 & 0.009 \\ per. & 0.002 & 0.990 & 0.008 \\ out & 0 & 0 & 1 \end{array}\right)$$

The state out describes the disappearance of banks due to failures and mergers.

The bulk of bilateral relationships and credit exposures in the interbank market involves banks of the core.

Links per block (number):

1297	5135
11533	1723

or

6.6%	26.1%
58.5%	8.8%

Credit exposures (EUR billion):

321.2	442.0
147.8	17.0

or

34.6%	47.6%
16.0%	1.8%

Banks of the periphery obtains 96% of the borrowing through EUR 442 billion in credit supplied by the core. Periphery provides EUR 147.8 billion to the core. The credit exchange between banks of the periphery is just 1.8% of the global credit volume exchanged.

By defining the credit volume intermediated by bank *i* as

$$\mathrm{Int}_{\mathrm{i}}=\mathrm{min}(\mathrm{borrowed}_{\mathrm{i}},\mathrm{lended}_{\mathrm{i}})$$

The credit volume intermediated by the banks of the core is 75% of the total credit volume intermediated. In fact

$$\frac{\sum_{i \in core} Int_i}{\sum_i Int_i} = 0.75$$

This specialization correlates with bank size but it is not only explained by the size attribute.

The relationships periphery banks maintain with core banks are in large extent exclusive and persistent over time.

63% of periphery banks borrow either from none or one single core bank (data are referring to the 2003 Q2 reference quarter but are representative of all the quarters). 96,7% of linkages that periphery banks use to borrow from core banks persist across quarters (91.4% on the lending side). If one does not condition on the credit side 93.2% of linkages present in one quarter are still present in the following one.

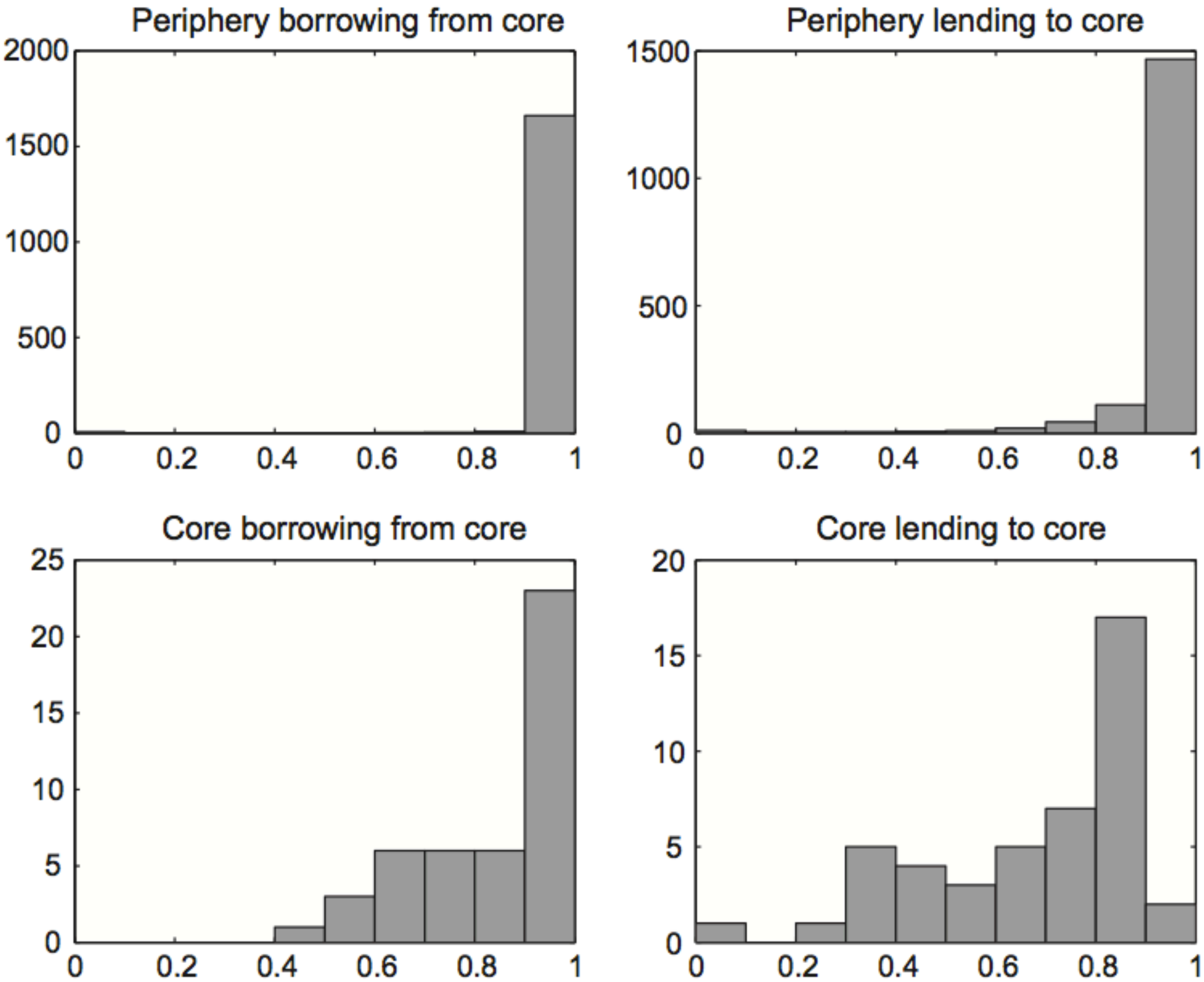


Fig. 4. Dependency on the core. The figure is based on a dependency ratio, the percentage of each bank's lending and borrowing (in value terms) that it directly transacts with core banks. Each histogram counts the number of banks who rely on core banks to the extent defined by the percentage brackets shown on the *x*-axis. The upper panels show periphery banks, the left panels focus on borrowing (the right panels on lending).

The first study on the interbank network had interpreted the degree distribution as compatible with a scale free distribution

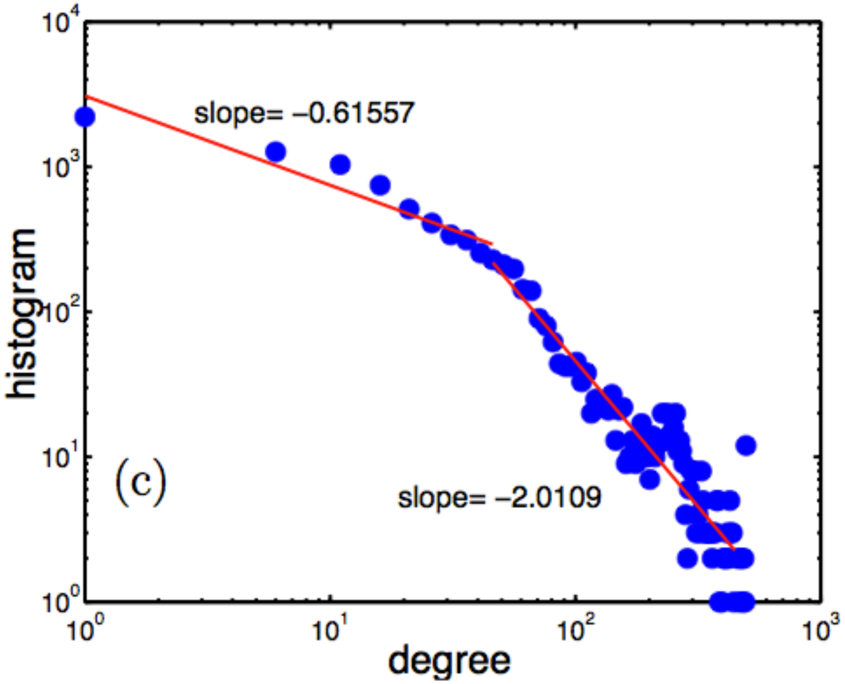


FIG. 3: Empirical out-degree (a) in-degree and (b) distribution of the interbank liability network. In (c) the degree distribution of the interbank connection network is shown. All the plots are histograms of aggregated data from all the 10 datasets.

Boss, M., Elsinger, H., Summer, M., & Thurner 4, S. (2004). Network topology of the interbank market. Quantitative Finance, 4(6), 677-684.

How the core-periphery model compare with basic models of random networks?

B. Craig, G. von Peter/J. Finan. Intermediation 23 (2014) 322–347

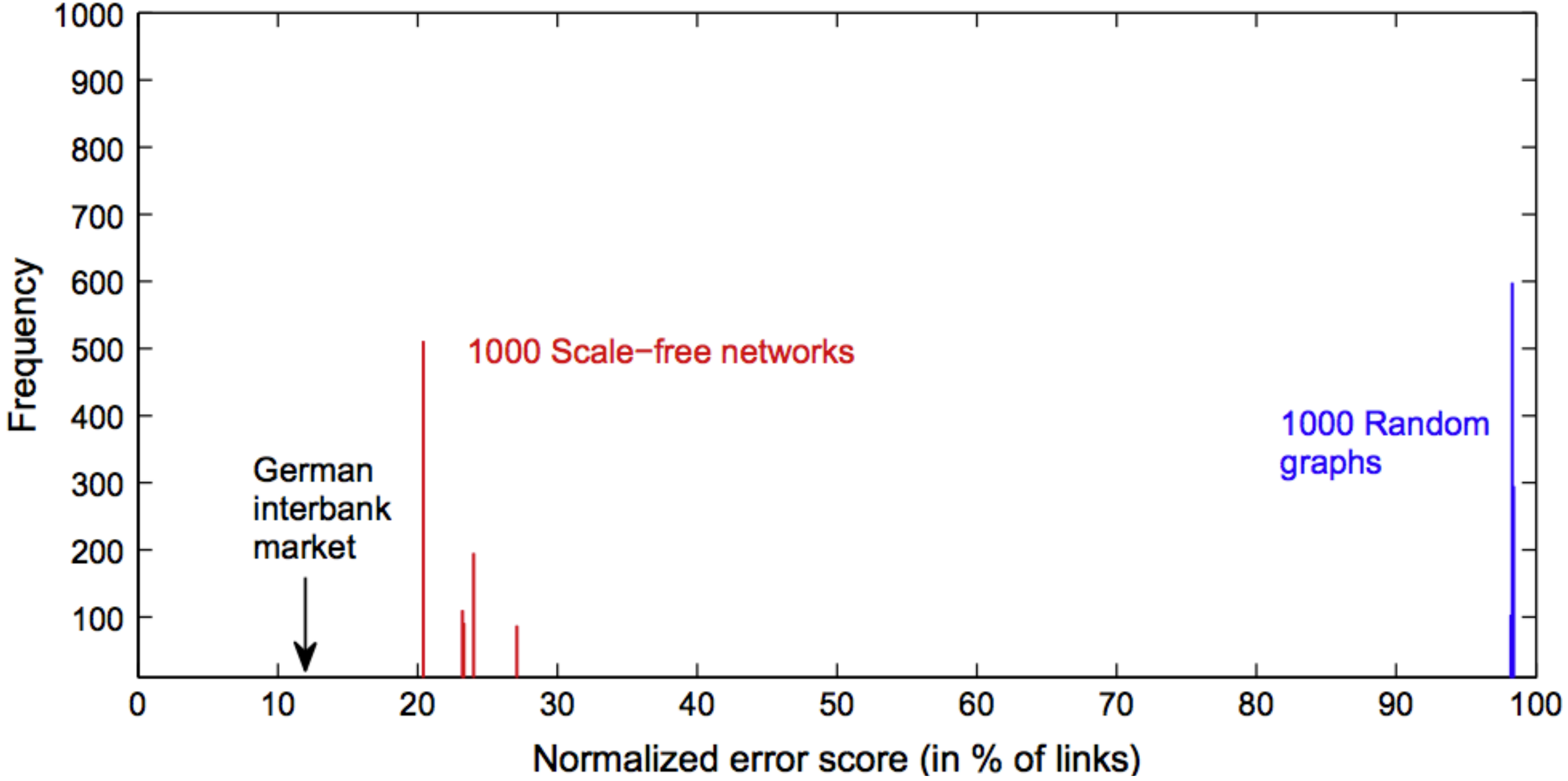


Fig. 7. German fit against simulated error score densities. This figure compares the total error score from fitting the tiering model to the German interbank network (12.2% of links, shown as an arrow) to the normalized error scores, as defined as in Eq. (3), from fitting two types of random networks of the same dimension. The red bars show the histogram of error scores from fitting 1000 scale-free networks, whereas the blue bars represent the histogram from fitting 1000 Erdös–Rényi random graphs. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

What characterize a core bank?

Craig and von Peter assembled bank variables for the 1802 active banks of the 2003 Q2 reference quarter.

These data were assembled by using monthly balance sheet data collected by the Bundesbank's statistics department.

These bank variables were used as regressors in a probit framework where the binary dependent variable is core membership defined as b_i=1 when the bank i is a core bank and b_i=0 otherwise.

$$\Pr\left(Y=1\middle|X\right)=\Phi\left(\alpha+X\beta\right)$$

where Φ is is the cumulative distribution function of the standard normal distribution.

Craig and von Peter investigate:

- observable measures of bank size (column 1 of the table of results),
- single network variables (column 2),
- measure of bank specialization (column 3).

The table shown in the next slide summarizes the results of the probit regression.

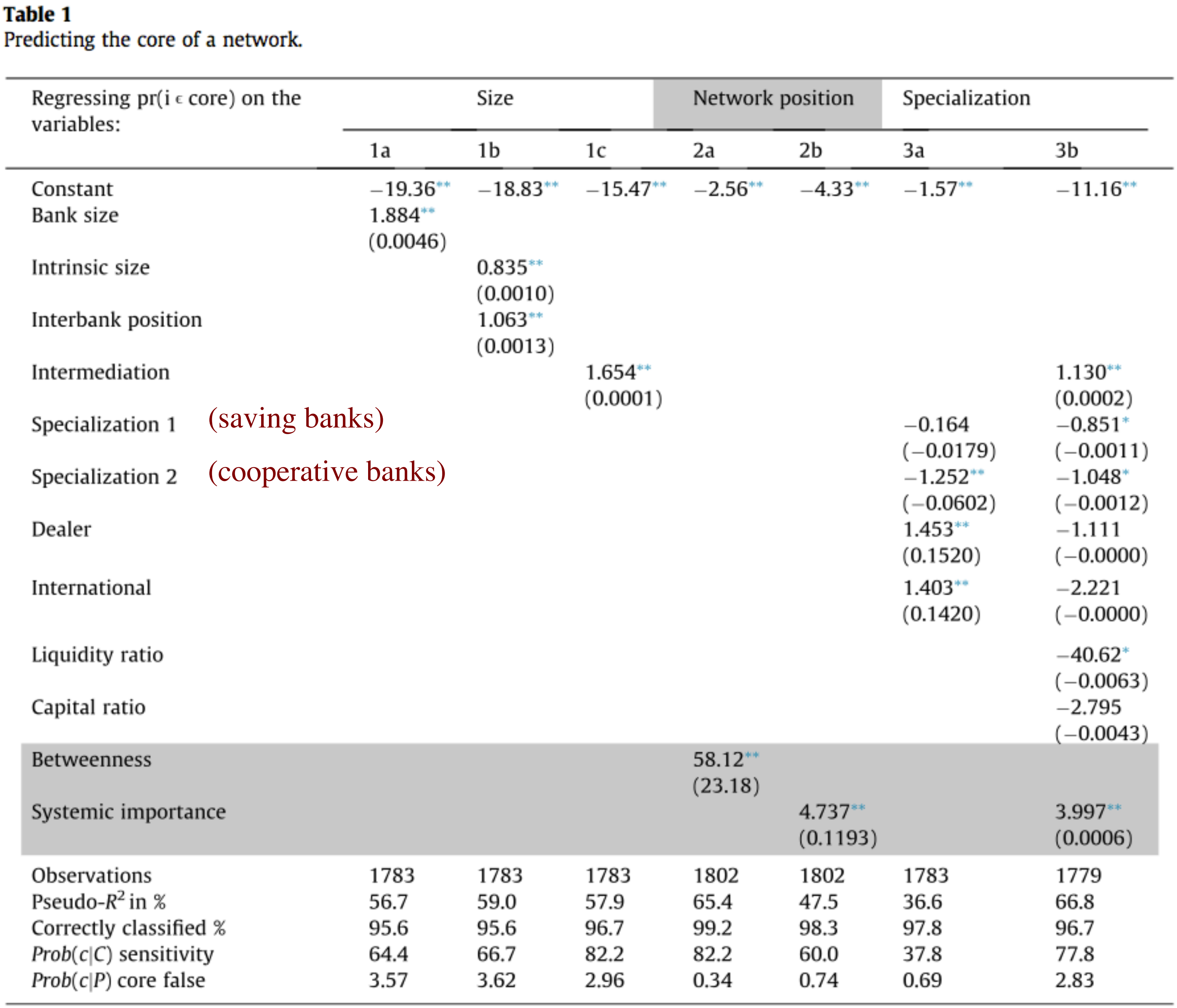


Table 1
Predicting the core of a network.

Regressing pr(i ϵ core) on the variables:	Size			Network position		Specialization	
	1a	1b	1c	2a	2b	3a	3b
Constant	−19.36**	−18.83**	−15.47**	−2.56**	−4.33**	−1.57**	−11.16**
Bank size	1.884** (0.0046)						
Intrinsic size		0.835** (0.0010)					
Interbank position		1.063** (0.0013)					
Intermediation			1.654** (0.0001)				1.130** (0.0002)
Specialization 1 (saving banks)						−0.164 (−0.0179)	−0.851* (−0.0011)
Specialization 2 (cooperative banks)						−1.252** (−0.0602)	−1.048* (−0.0012)
Dealer						1.453** (0.1520)	−1.111 (−0.0000)
International						1.403** (0.1420)	−2.221 (−0.0000)
Liquidity ratio							−40.62* (−0.0063)
Capital ratio							−2.795 (−0.0043)
Betweenness				58.12** (23.18)			
Systemic importance					4.737** (0.1193)		3.997** (0.0006)
Observations	1783	1783	1783	1802	1802	1783	1779
Pseudo-R^2 in %	56.7	59.0	57.9	65.4	47.5	36.6	66.8
Correctly classified %	95.6	95.6	96.7	99.2	98.3	97.8	96.7
Prob(*c*|*C*) sensitivity	64.4	66.7	82.2	82.2	60.0	37.8	77.8
Prob(*c*|*P*) core false	3.57	3.62	2.96	0.34	0.74	0.69	2.83

F

The results of the previous table show that network position is predictable from bank-specific features.

Core banks are active at national and international level and provide a broader range of financial services than local or specialized banks.

Core banks carry out large transactions and perform a large fraction of intermediation. Craig and von Peter conclude that "Banks in the core .... can therefore be regarded as *money center banks*", i.e. large banks dominating wholesale activity in money markets. These banks often provide clearing services and corresponding banking services and act as dealers in a broad range of markets.

In several cases bilateral positions are unknown whereas aggregate positions of some (economic) actors are reported.

This is quite common in the case of the Interbank market.

From balance-sheet the total interbank assets A_i and liabilities L_i of each bank *i* are known whereas bilateral positions are unknown (in some cases also to their national banks).

The standard approach in the economic literature is to estimate the bilateral positions by using the method of maximum entropy.

The assumption underlying this method is that each economic actor is looking for the maximal diversification of its credit relationships and maximal spreading of the credit relationships is provided by the maximum entropy solution.

Maximum entropy state is in fact the state associated with the minimum specification of available information.

Let us call X_{ij} the elements of the matrix representing gross interbank positions. For each bank *i*

Interbank assets: $A_i = \sum_{j=1}^{N} X_{ij}$

Interbank liabilities: $L_i = \sum_{j=1}^{N} X_{ji}$

where the elements X_{ij} of the matrix **X** represent the amount bank *i* lends to bank *j*

In the maximum entropy method, a solution E_{ij} is obtained for a proxy of X_{ij} by minimizing the entropy function

$$\sum_{ij} E_{ij} \ln\left(\frac{E_{ij}}{Q_{ij}}\right)$$

under the constraints

$$A_i = \sum_{j=1}^{N} E_{ij}$$

where Q_{ij} are known bilateral exposures (when available)

In the simplest case when only the marginal distributions are known and positive, the solution will be close to the approximation

$$E_{ij} \propto A_i L_j$$

note that this implies that the associated minimum entropy network is typically a fully connected network (although a weighted one).

Empirical observations show that the proxy obtained by maximum entropy is quite different from what observed in reality. In fact when the network of interbank positions is monitored the detected empirical network is a sparse network and not a fully connected one.

Recently Anand, Craig, and von Peter have proposed an alternative method to estimate the bilateral positions. They call their method the minimum density (MD) method.

The economic assumption underlying this new method is that there is a fixed cost c to establish a credit relationship between bank i and bank j. This assumption implies that the network will result from the action each bank will take to minimize their costs.

K. Anand, B. Craig, G. von Peter, Filling in the blanks: network structure and interbank contagion. Quantitative Finance (in press 2015), BIS working papers n. 455.

The minimum density approach can be formulated as a constrained optimization problem for the matrix **Z**

$$\min_{Z} c \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{1}_{\left[Z_{ij}>0\right]}$$

$$\sum_{j=1}^{N} Z_{ij} = A_i \qquad \forall i = 1,2,\ldots,N$$

$$\sum_{i=1}^{N} Z_{ij} = L_j \qquad \forall j = 1,2,\ldots,N$$

$$Z_{ij} > 0 \qquad \forall i,j$$

The optimization problem is performed by assigning penalties for deviations observed from the marginals. The following quantities are used during the optimization process.

$$AD_i \equiv \left(A_i - \sum_j Z_{ij} \right)$$

$$LD_i \equiv \left(L_i - \sum_j Z_{ji} \right)$$

The optimization process maximizes the objective function

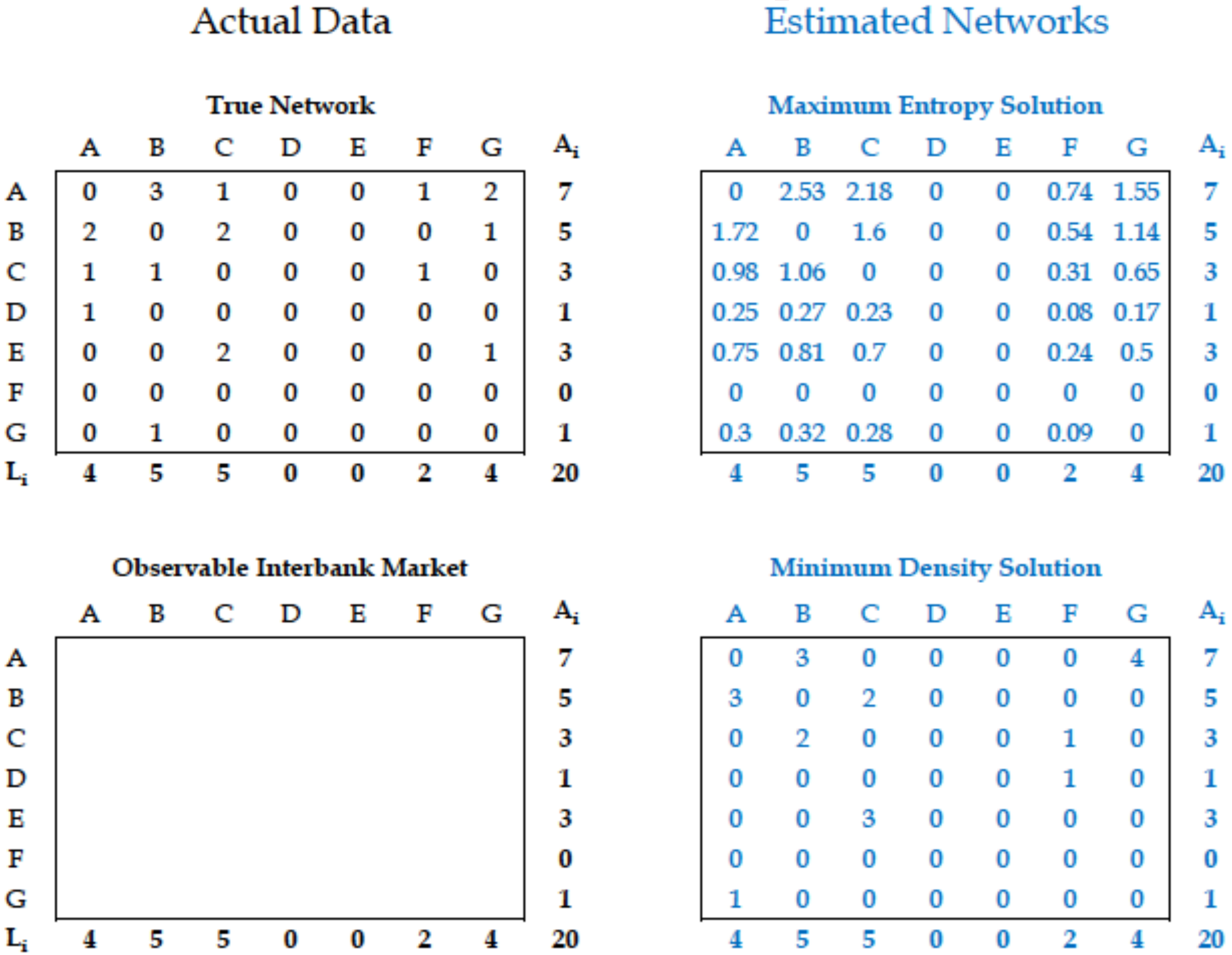
$$V(\mathbf{Z}) = -c \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{1}_{\left[Z_{ij}>0\right]} - \sum_{i=1}^{N} \left[\alpha_i AD_i^2 + \delta_i LD_i^2 \right]$$

where α_i and δ_i are optimizing parameters.

Logical steps of the optimizing algorithm:

- at each iteration a link (i,j) is selected with probability P_{ij} proportional to $\max\{AD_i/LD_J,LD_j/AD_i\}$.
- the exposure Z_{ij} is loaded with the maximum value that the (i,j) pair of banks can transact given their current asset and liability positions, i.e., $Z_{ij}=\min\{AD_i,LD_j\}$.
- if adding this link the value of the objective function increases $V(Z_{in})>V(Z)$ the allocation is retained.
- in case the objective value function decreases the link is retained with probability $P(Z_{in})$ prop. to $\exp[V(Z_{in})-V(Z)]$.
- otherwise the link is rejected.
- the process is iterated until the total interbank market has been allocated.

An illustrative example



Actual Data

True Network

	A	B	C	D	E	F	G	A_i
A	0	3	1	0	0	1	2	7
B	2	0	2	0	0	0	1	5
C	1	1	0	0	0	1	0	3
D	1	0	0	0	0	0	0	1
E	0	0	2	0	0	0	1	3
F	0	0	0	0	0	0	0	0
G	0	1	0	0	0	0	0	1
L_i	4	5	5	0	0	2	4	20

Observable Interbank Market

	A	B	C	D	E	F	G	A_i
A								7
B								5
C								3
D								1
E								3
F								0
G								1
L_i	4	5	5	0	0	2	4	20

Estimated Networks

Maximum Entropy Solution

	A	B	C	D	E	F	G	A_i
	0	2.53	2.18	0	0	0.74	1.55	7
	1.72	0	1.6	0	0	0.54	1.14	5
	0.98	1.06	0	0	0	0.31	0.65	3
	0.25	0.27	0.23	0	0	0.08	0.17	1
	0.75	0.81	0.7	0	0	0.24	0.5	3
	0	0	0	0	0	0	0	0
	0.3	0.32	0.28	0	0	0.09	0	1
	4	5	5	0	0	2	4	20

Minimum Density Solution

	A	B	C	D	E	F	G	A_i
	0	3	0	0	0	0	4	7
	3	0	2	0	0	0	0	5
	0	2	0	0	0	1	0	3
	0	0	0	0	0	1	0	1
	0	0	3	0	0	0	0	3
	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	1
	4	5	5	0	0	2	4	20

The case of interbank positions in Germany

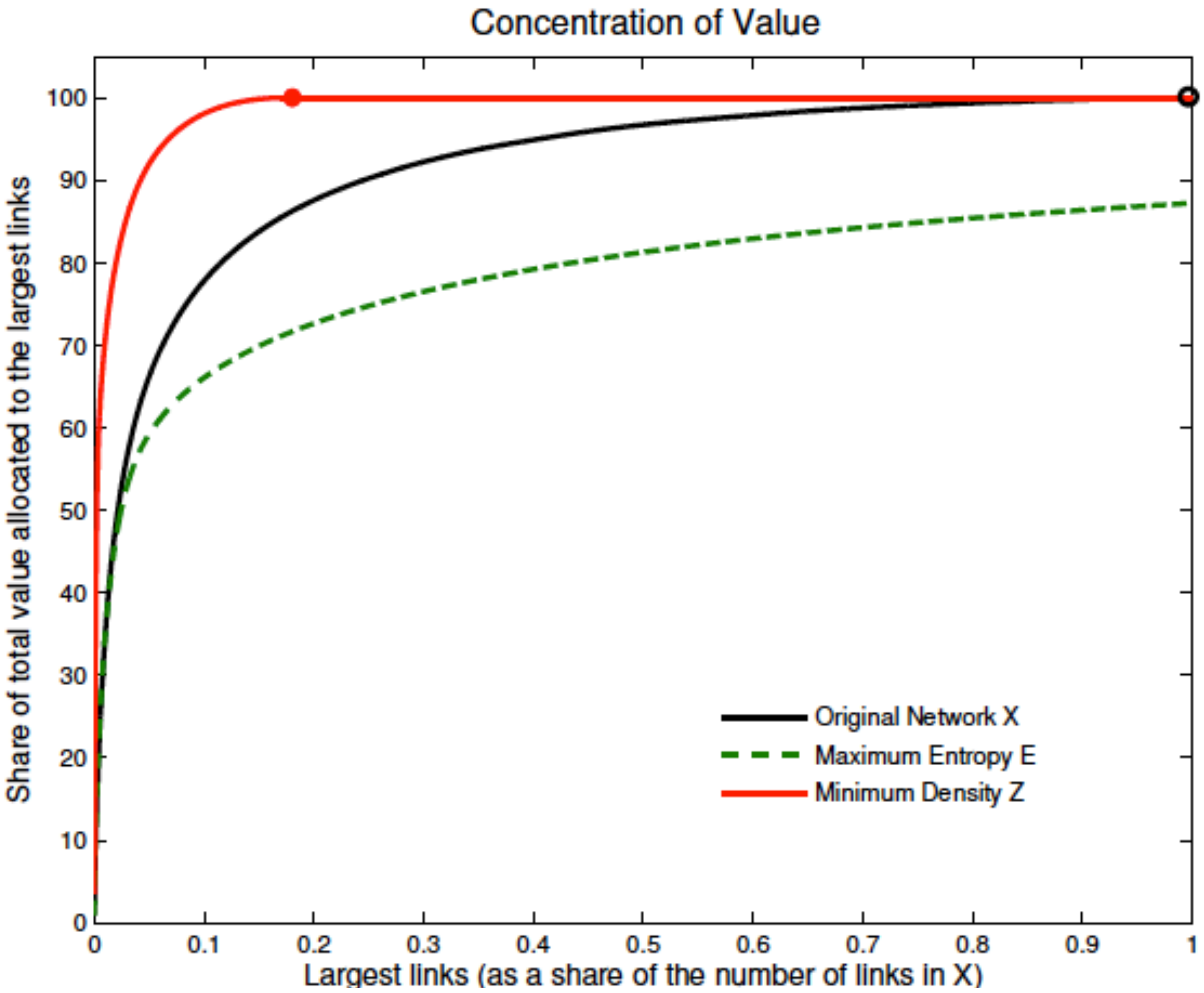


Figure 2: The figure shows the concentration of value on the largest links for the different networks. The x-axis ranks bilateral linkages (in descending order of size) and expresses the first n links as a share of the total number of links in the original network $\mathbb{X}$ (18,624). The y-axis shows the cumulative share of value allocated to the largest n links, relative to the total interbank volume. The dots indicate at which point 100% of volume has been reached. For $\mathbb{X}$ this is at unity, for $\mathbb{Z}$ this occurs at 0.185, whereas $\mathbb{E}$ needs 158 times the number of links in $\mathbb{X}$ before reaching 100% of interbank volume.

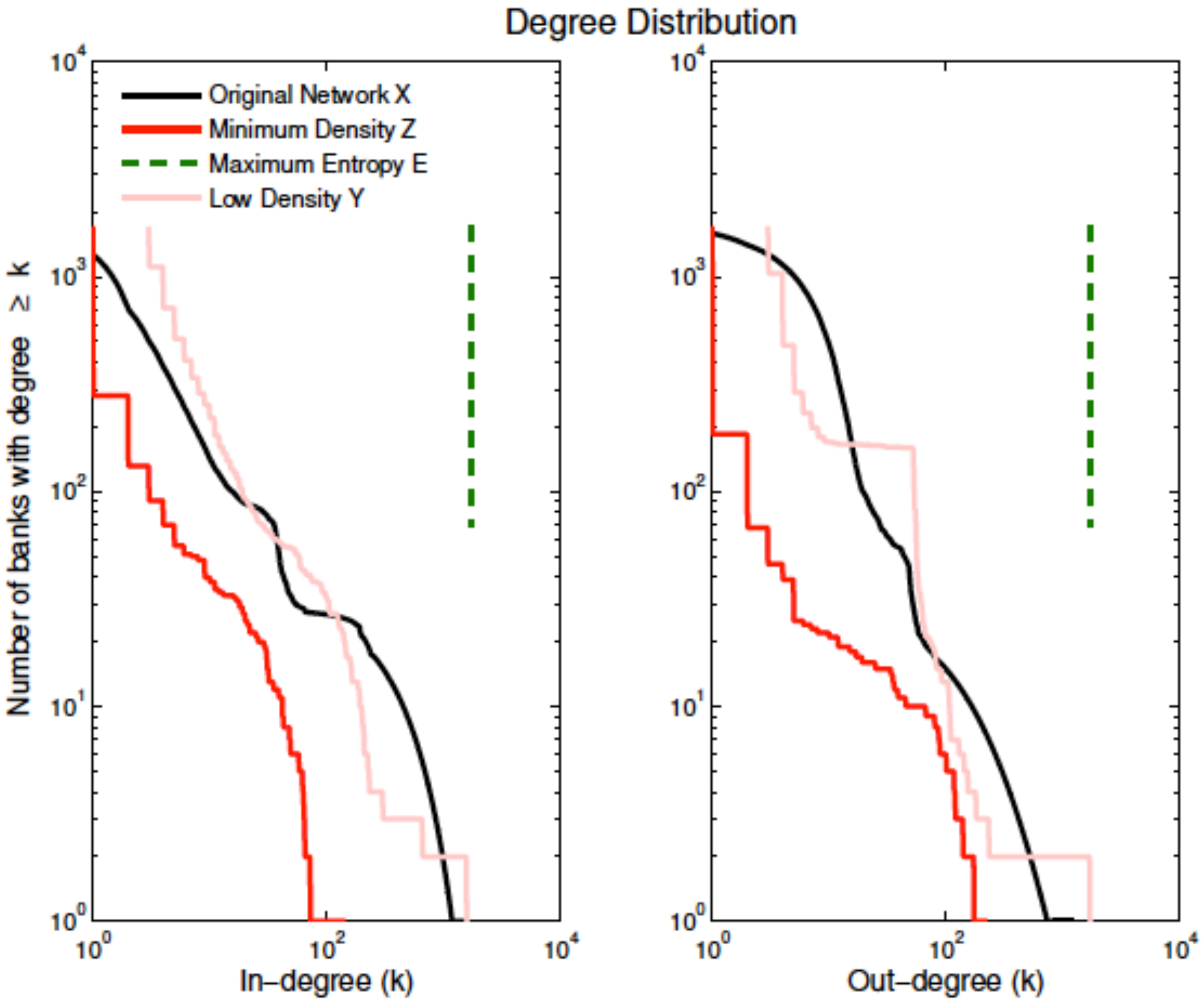
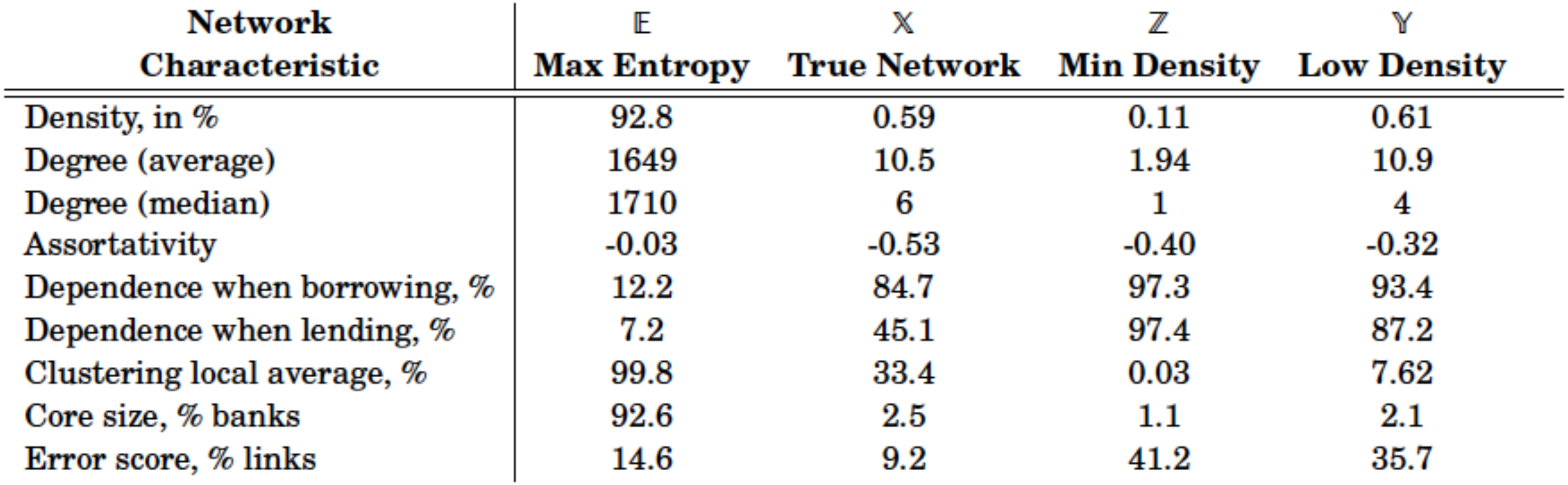


Figure 3: The figure displays the degree distribution in its cumulative form, showing the number of banks with a degree greater than the number shown on the x-axis, on a double log scale. A straight line would indicate a Pareto cumulative indicative of a power law distribution. The degree distribution of the original network $\mathbb{X}$ has been smoothed to preserve the confidentiality of individual bank data, and shows averages at the end points, instead.

Network metrics



Network Characteristic	$\mathbb{E}$ Max Entropy	$\mathbb{X}$ True Network	$\mathbb{Z}$ Min Density	$\mathbb{Y}$ Low Density
Density, in %	92.8	0.59	0.11	0.61
Degree (average)	1649	10.5	1.94	10.9
Degree (median)	1710	6	1	4
Assortativity	-0.03	-0.53	-0.40	-0.32
Dependence when borrowing, %	12.2	84.7	97.3	93.4
Dependence when lending, %	7.2	45.1	97.4	87.2
Clustering local average, %	99.8	33.4	0.03	7.62
Core size, % banks	92.6	2.5	1.1	2.1
Error score, % links	14.6	9.2	41.2	35.7

Table 1: Comparing basic network features of benchmark estimates with those of the original German interbank network.

In the case of the low density the second step of the algorithm is modified as

- the exposure Z_{ij} is loaded with the maximum value that the (i,j) pair of banks can transact given their current asset and liability positions, i.e., $Z_{ij}=\lambda \min\{AD_i,LD_j\}$

with $\lambda<1$

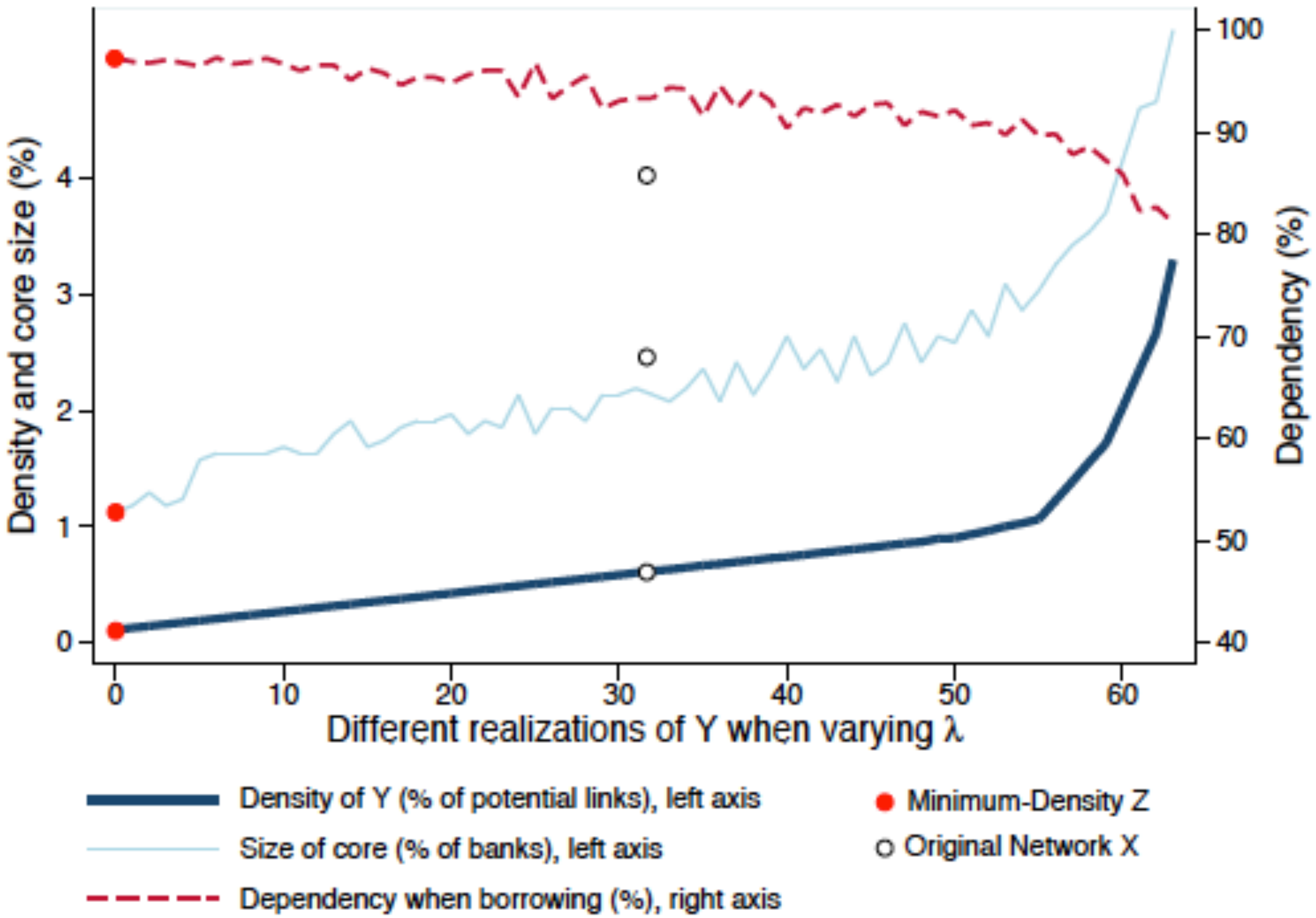


Figure 4: This figure shows three network features for 65 different low-density solutions $\mathbb{Y}$. The implementation here sets $\lambda = 0.5$ for the first k links being filled by the algorithm and $\lambda = 1$ thereafter, with k raised from 0 to 100,000 in 65 (unequally spaced) steps. The first realization (at $k = 0$) is the MD network $\mathbb{Z}$ with the network features shown as red dots (as in Table 1). The black circles indicate the values for the original network $\mathbb{X}$, plotted at the point where a comparable low-density network $\mathbb{Y}$ reaches a density similar to $\mathbb{X}$ (at k=16,000).

Interbank market: a study of the e-MID database and an agent based model

- electronic transactions between 254 Italian banks;

- transactions are transparent;

- time period from January 1999 to December 2009;

- information about the "aggressor" (lender or borrower);

- overnight and overnight-long credit relationships;

- data analyzed in 3-maintainance periods (1-maintainance period is about 23 trading days usually close to q1 calendar month).

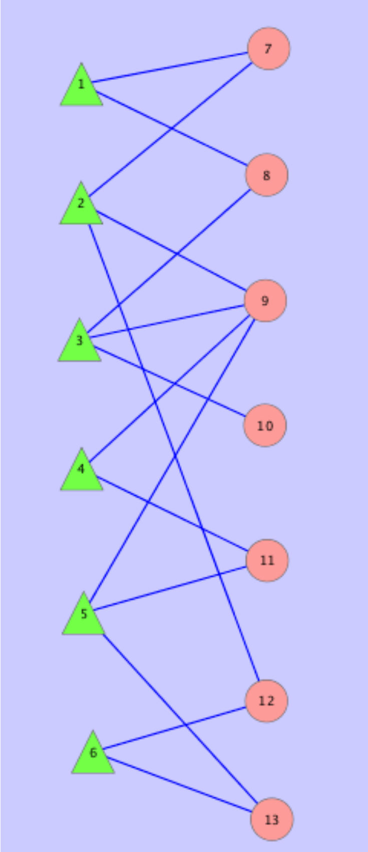
¶Hatzopoulos, V., Iori, G., Mantegna, R. N., Miccichè, S., & Tumminello, M. (2015). Quantifying preferential trading in the e-MID interbank market. *Quantitative Finance*, *15*(4), 693-710.

¶Iori, G., Mantegna, R. N., Marotta, L., Miccichè, S., Porter, J., & Tumminello, M. (2015). Networked relationships in the e-MID Interbank market: A trading model with memory. Journal of Economic Dynamics and Control, 50, 98-116.

Empirical analyses[¶] of the e-MID database show evidence of the networked nature of the interbank market

Lender aggressor or borrower aggressor transactions

Lending banks

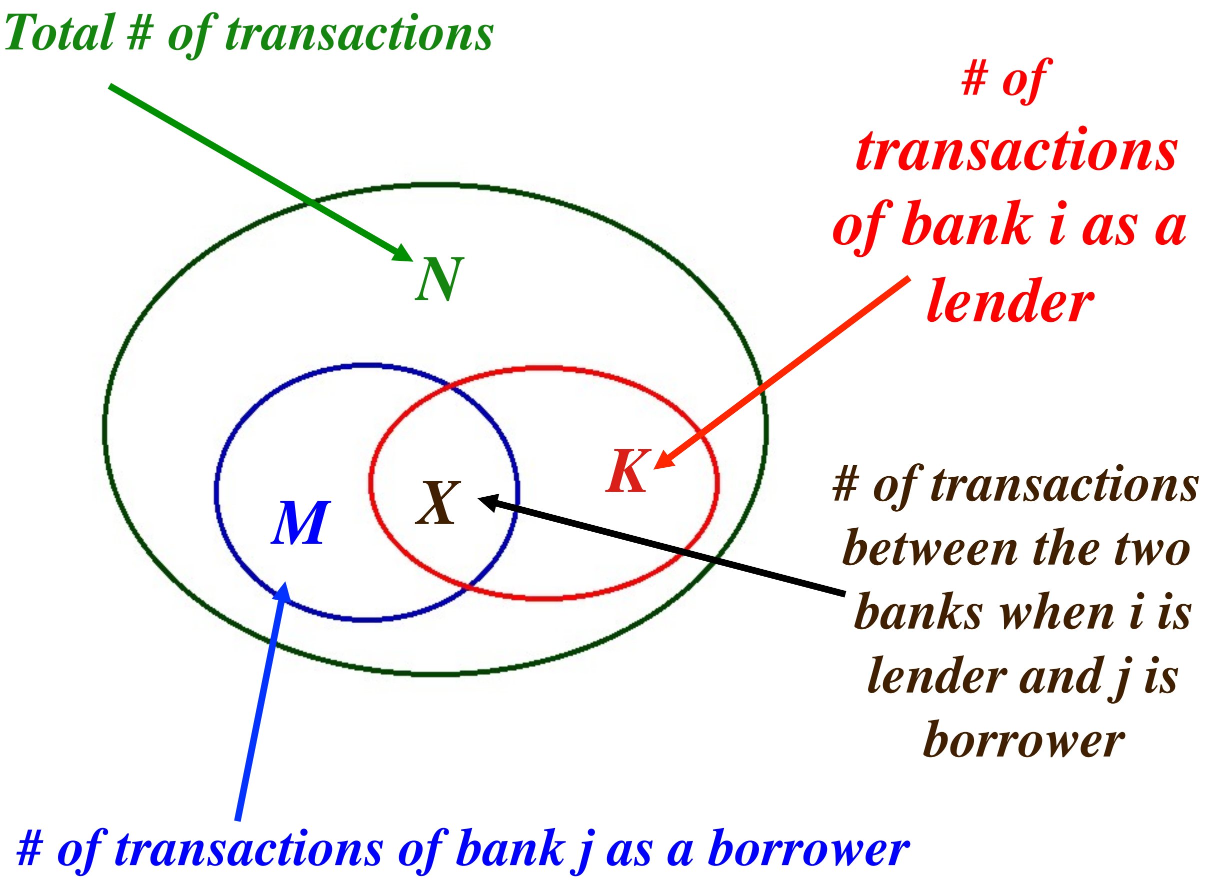


Borrowing banks

We performed a statistical validation of the over-expression and under-expression of repeated credit transactions.

[¶] Hatzopoulos, V., Iori, G., Mantegna, R. N., Miccichè, S., & Tumminello, M. (2015). Quantifying preferential trading in the e-MID interbank market. *Quantitative Finance*, *15*(4), 693-710.

We statistically validate the relationships between banks i (***lender***) and j (***borrower***) over a given time period.



Total # of transactions

# of transactions of bank i as a lender

# of transactions between the two banks when i is lender and j is borrower

# of transactions of bank j as a borrower

What is the probability P(X | N,M,K)?

Hypergeometric distribution

$$P(X \mid N,M,K) = \frac{\binom{M}{X}\binom{N-M}{K-X}}{\binom{N}{K}}$$

p-value

OVER-expression:

$$p = 1 - \sum_{i=0}^{X-1} \frac{\binom{M}{i}\binom{N-M}{K-i}}{\binom{N}{K}}$$

UNDER-expression:

$$p = \sum_{i=0}^{X} \frac{\binom{M}{i}\binom{N-M}{K-i}}{\binom{N}{K}}$$

threshold θ*: 5%, 1%, …*

Multiple hypothesis test correction is needed

in order to control false positives expected in multiple comparisons

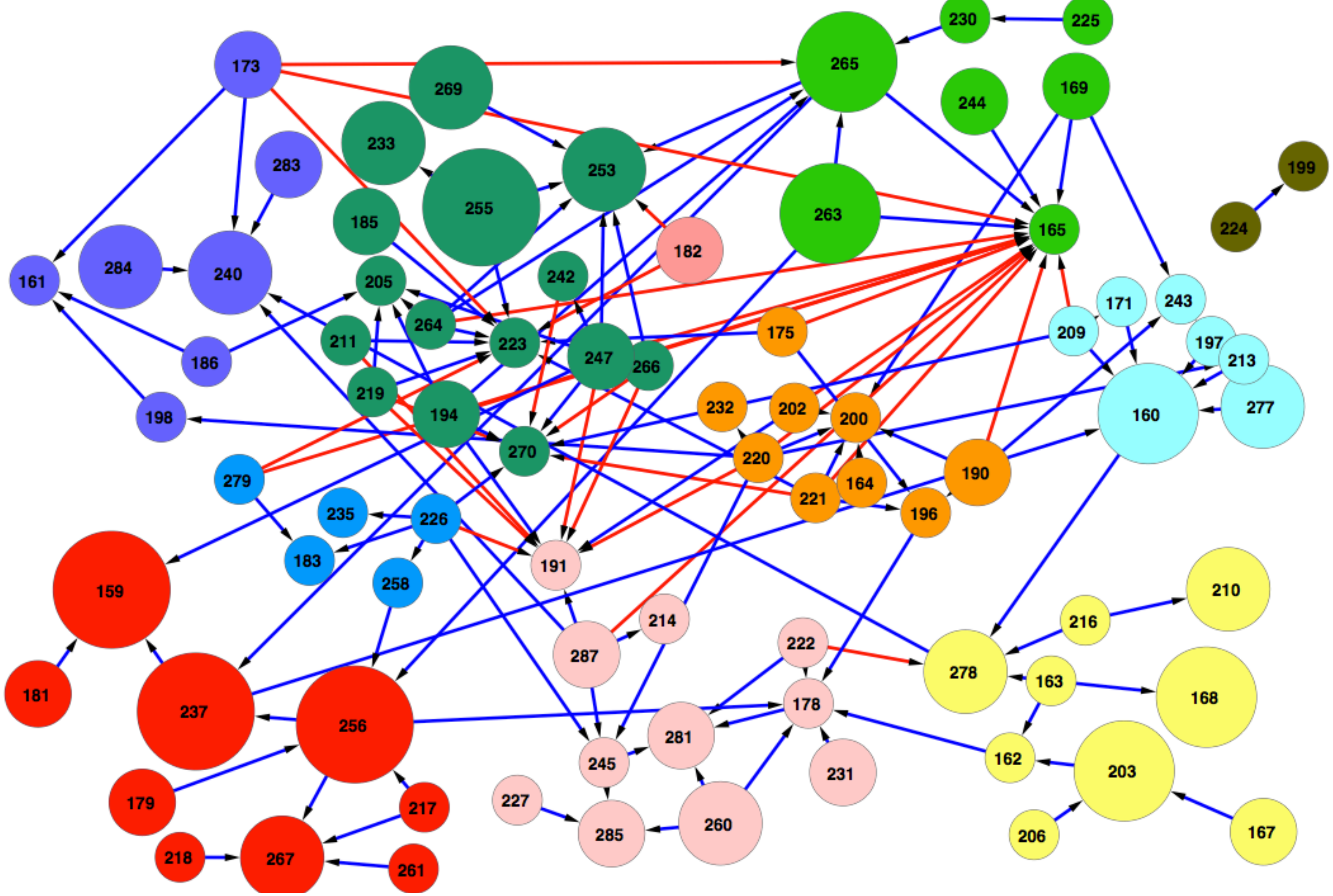
Bonferroni correction

The threshold θ *must be divided by the number* N_t *of performed tests:* $p_B < \theta / N_t$

False Discovery Rate correction

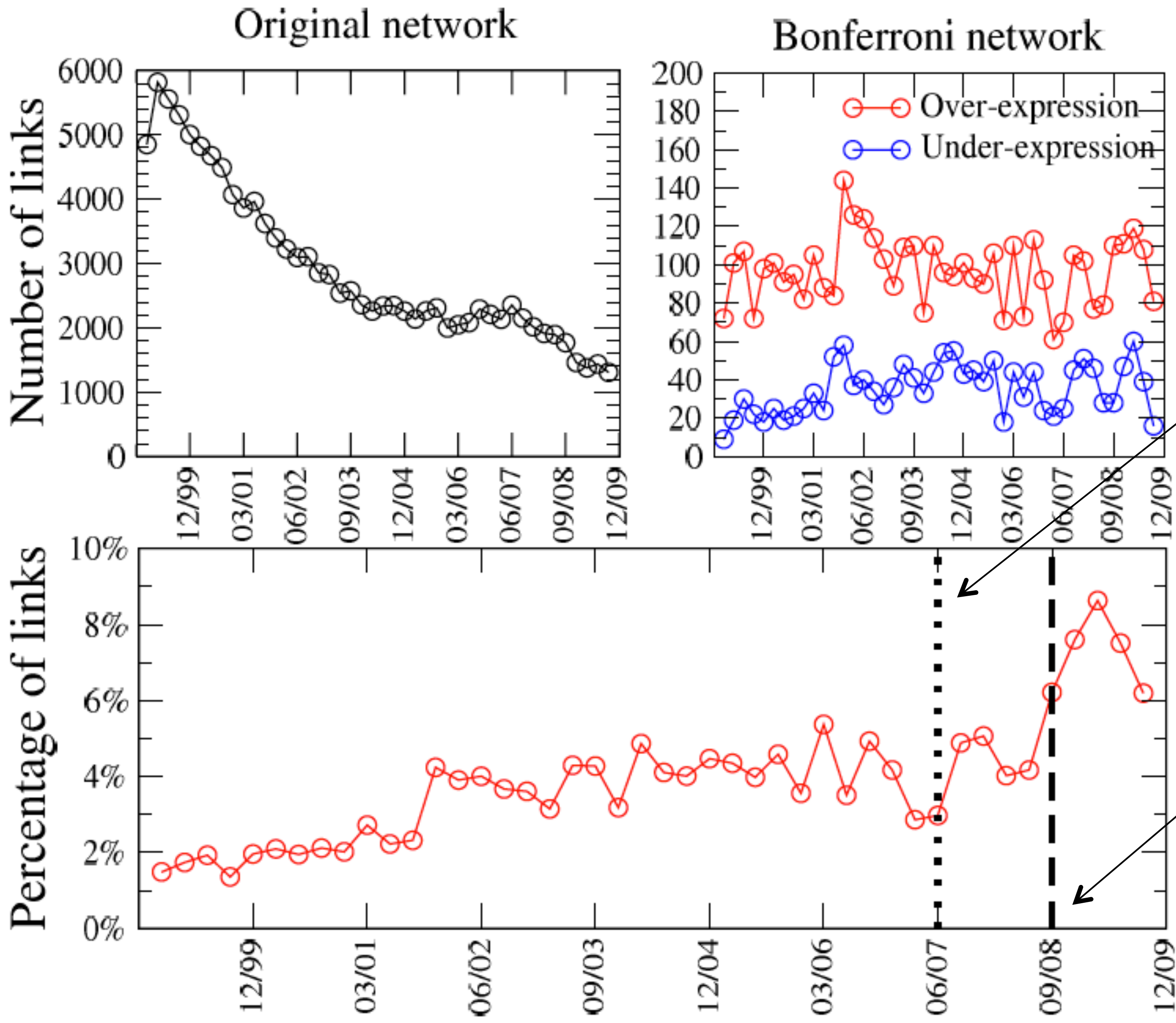
$P_1 < \theta/N_t$
$P_2 < 2\ \theta/N_t$
$P_3 < 3\ \theta/N_t$
…
$P_k < k\ \theta/N_t$

Lender aggressor transactions. Bonferroni network of the 3-maintenance period 10-Sep-2008 / 09-Dec-2008



The different colors indicate the node membership to the partitions detected by using the Radatool algorithm (unweighted option). Red links are under-expressed links, while blue links are over-expressed ones.

Lender-aggressor dataset



Freezing (or stressing) of the interbank market

Lehman's bankruptcy.

A model[¶] for over-expressed bilateral credit relationships

Underlying idea: two banks are more likely to have a credit relationship if they already had one(s) in the recent past.

Separate (but parallel) modeling of the lender-aggressor and borrower-aggressor case.

Calibration of the model parameters on real data.

[¶] Iori, G., Mantegna, R. N., Marotta, L., Miccichè, S., Porter, J., & Tumminello, M. (2015). Networked relationships in the e-MID Interbank market: A trading model with memory. Journal of Economic Dynamics and Control, 50, 98-116.

An order to borrow is placed by a bank B_i that is randomly selected with probability:

$p_b(B_i,t)$ prop to $B_i^{b,la}(t)$

Quantity $B_i^{b,la}(t)$ is the number of transactions that bank B_i has planned to do in a time-window T_M, as a borrower, in the lender-aggressor transactions after t transactions already occurred in T_M.

A lender is then selected with probability

$$p_b\left(B_j,t\middle|B_i\right) \propto B_j^{l,la}(t)\left[w + N\left(B_j,B_i,t\right)\right]$$

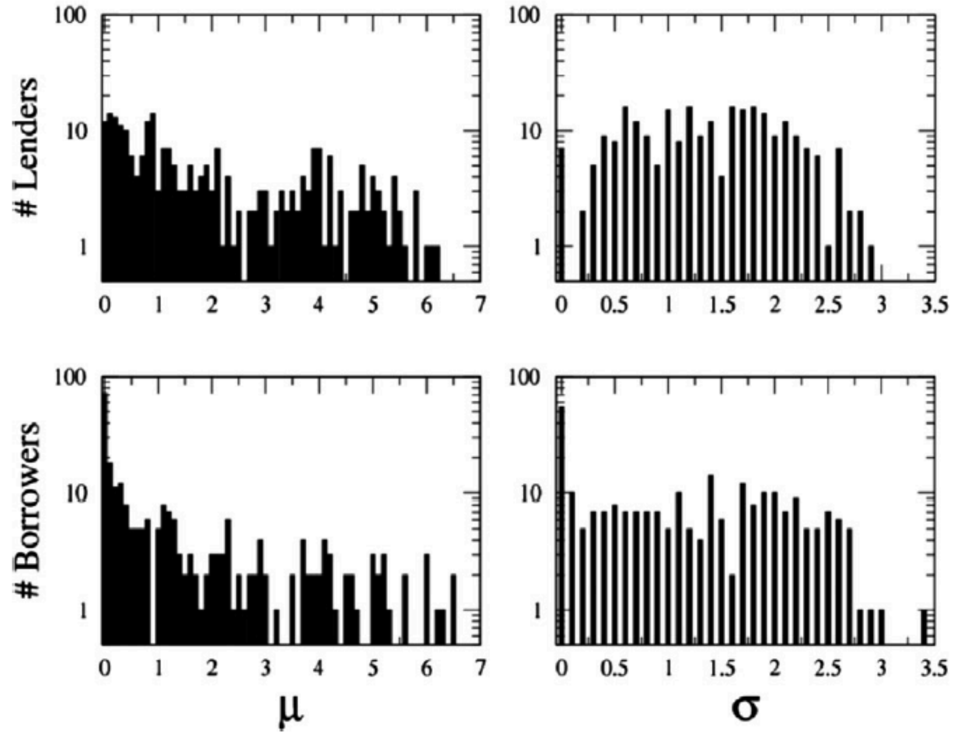
Where $B_j^{l,la}(t)$ is the number of transactions that bank B_j has planned to do in a time-window T_M, as a lender after t transactions already occurred in T_M . $N(B_j,B_i,t)$ is the total number of transactions in which B_j lent money to B_i over the past Q time-windows, in spite of the type of transaction. w is a parameter, assumed to be equal for all the banks, which represents a common level of attractiveness of borrowers.

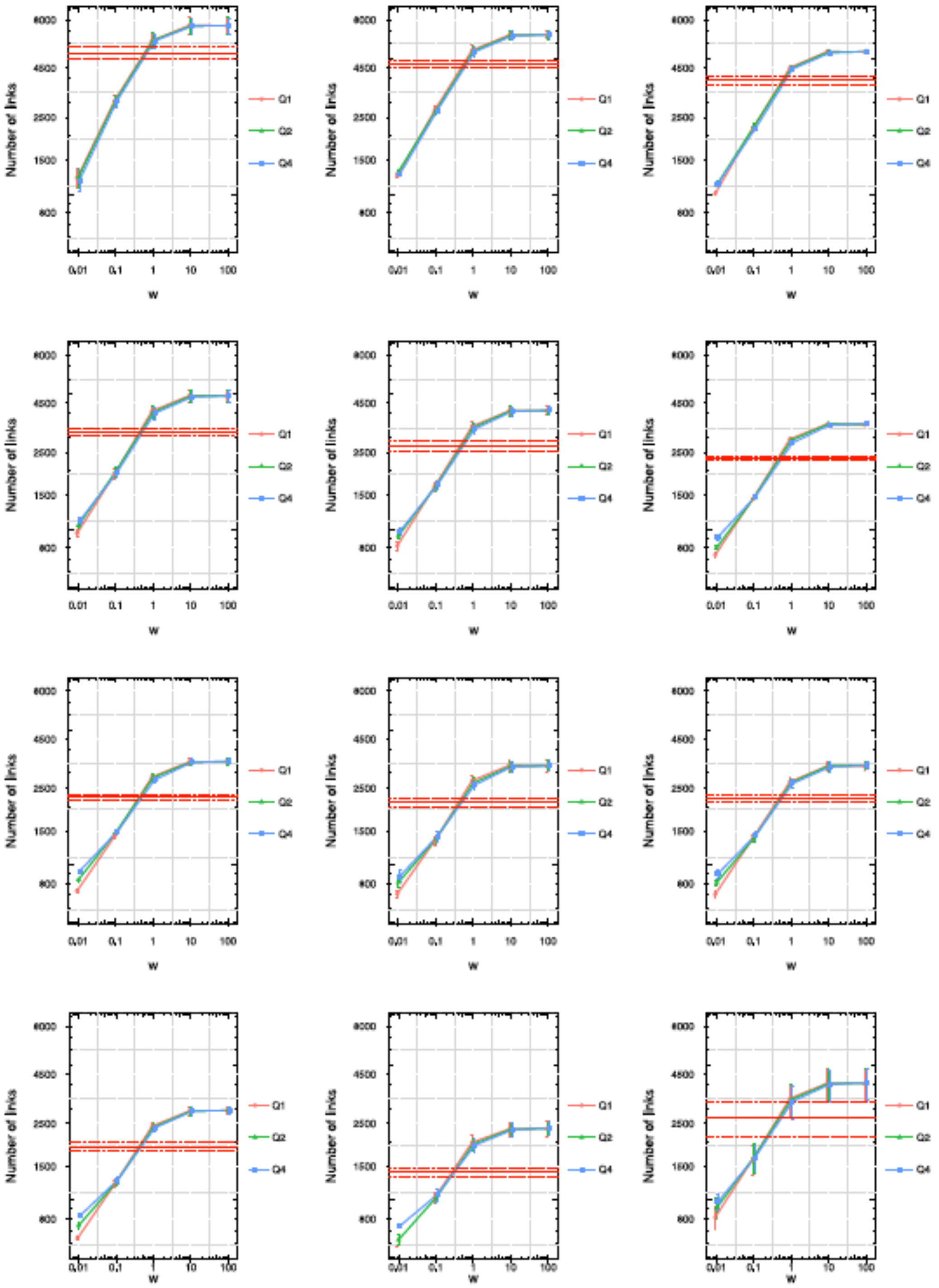
The probability that a lender-aggressor transaction occurs from *j* (the lender) to *i* (the borrower) after *t* transactions is

$$p\left(B_j,B_i,t\right)=p_b\left(B_i,t\right)p_l\left(B_j,t\middle|B_i\right)=$$

$$=\frac{B_i^{b,la}\left(t\right)}{\sum_{k=1}^{n}B_k^{b,la}\left(t\right)}\frac{B_j^{l,la}\left(t\right)\left[w+N\left(B_j,B_i,t\right)\right]}{\sum_{q=1}^{n}B_q^{l,la}\left(t\right)\left[w+N\left(B_j,B_i,t\right)\right]}$$

The distribution of $B^{l,la}$ and $B^{b,la}$ was assumed to be Lognormal and the parameters μ and σ of the Lognormal distribution were estimated from empirical data





Calibration of the *w* parameter from the empirical data (number of links)

Log–log plot of the number of links in lender-aggressor simulated networks, for different values of *w* and *Q*, and real network (solid red line). The comparison is done separately for each year by averaging over 4 consecutive 3-maintenance periods. The last plot (bottom, right) shows the overall result when averaging over the periods 11–30. Dotted red lines indicate the intervals at plus or minus one standard deviation.

With fine tuning, *w* is ranging from 1 to 0.1. A good global compromise is *w*=1

Calibration of the *Q* parameter from the empirical data when setting *w*=1 (lagged Jaccard index between networks of real data and simulations).

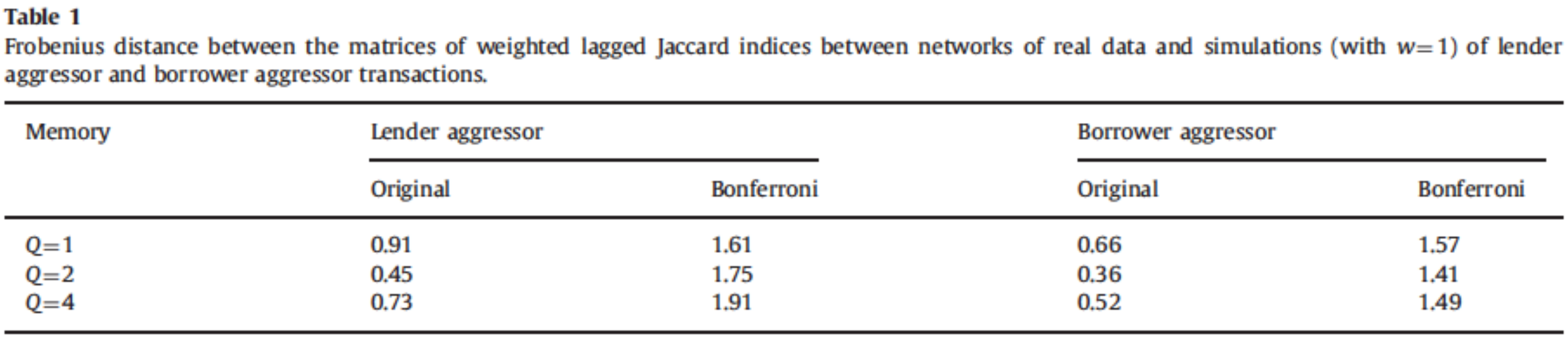
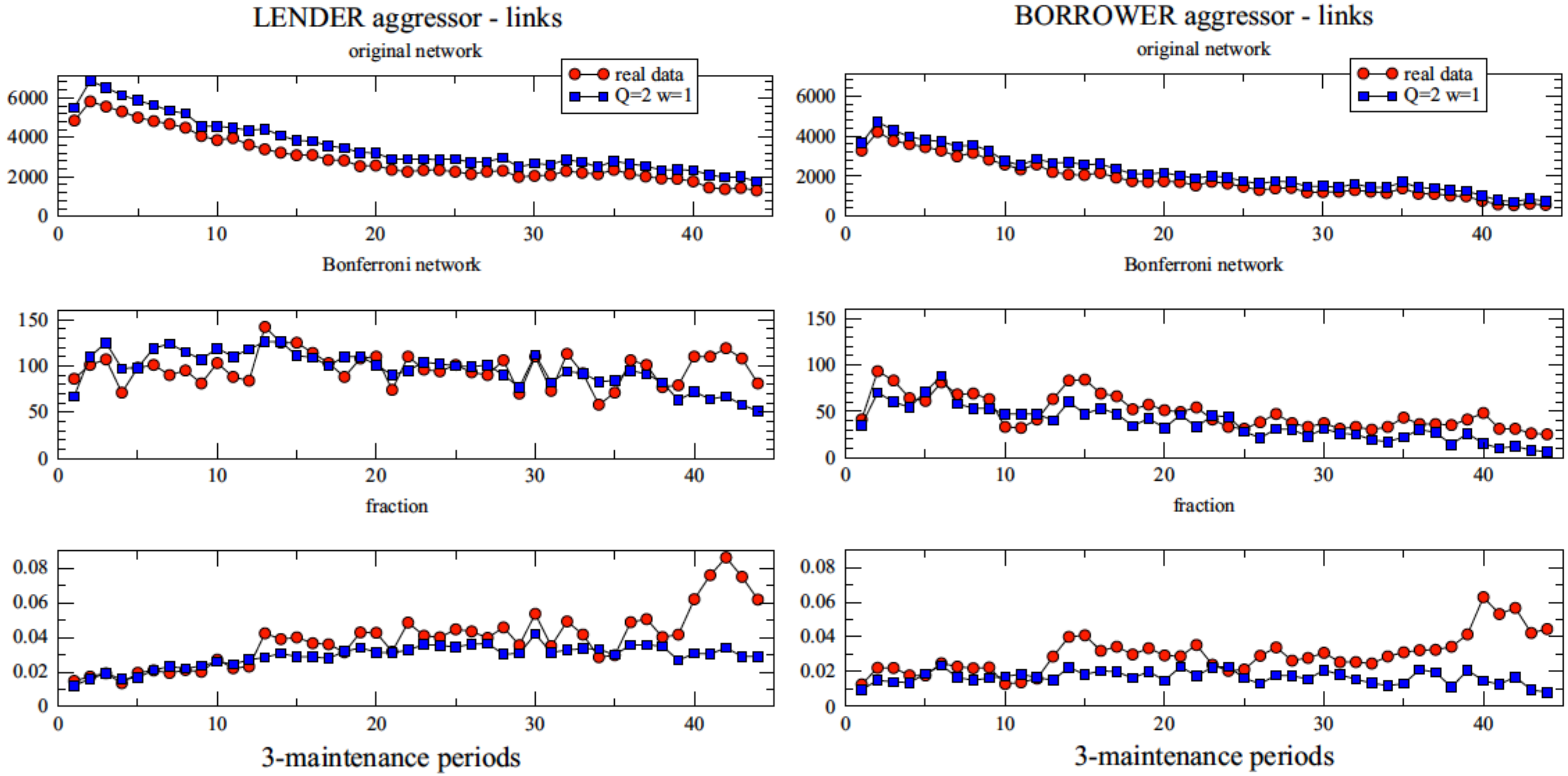


Table 1
Frobenius distance between the matrices of weighted lagged Jaccard indices between networks of real data and simulations (with $w=1$) of lender aggressor and borrower aggressor transactions.

Memory	Lender aggressor		Borrower aggressor	
	Original	Bonferroni	Original	Bonferroni
$Q=1$	0.91	1.61	0.66	1.57
$Q=2$	0.45	1.75	0.36	1.41
$Q=4$	0.73	1.91	0.52	1.49

The best choice is Q=2

Results of the simulation after calibration (w=1, Q=2) and comparison with empirical data



We quantify the similarity of two networks by using the weighted Jaccard index between networks, net_1 and net_2,

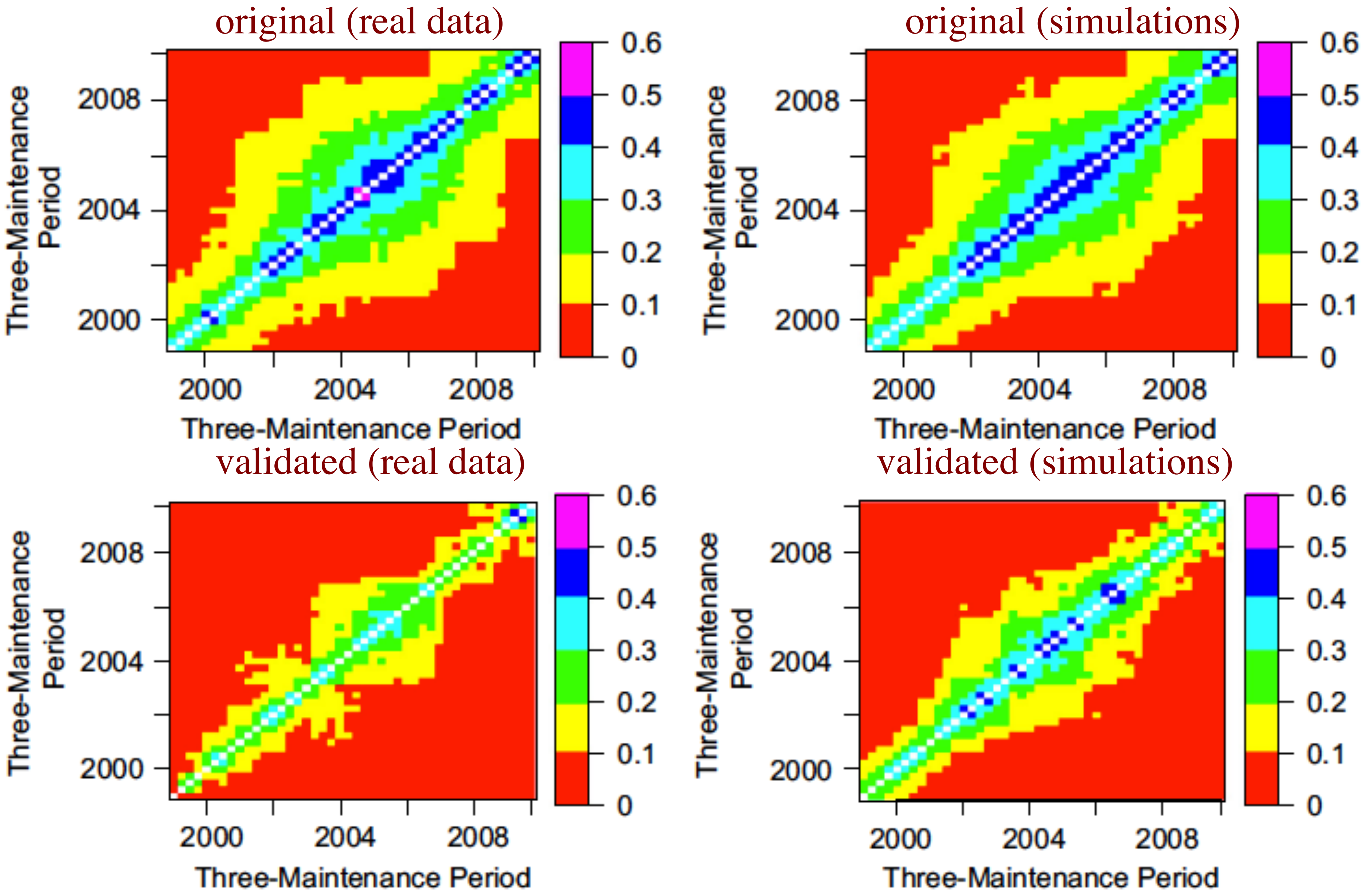
$$J_W\left(net_1, net_2\right) = \frac{\sum_{i,j} \min\left[w_{i,j}^1, w_{i,j}^2\right]}{\sum_{i,j} \max\left[w_{i,j}^1, w_{i,j}^2\right]}$$

where $w^1_{i,j}$ is the weight of link i,j in the first network, net_1, and $w^2_{i,j}$ is the weight of link i,j in the second network, net_2.

For an unweighted network $J_W(net_1,net_2)$ reduces to the usual Jaccard index

$$J_W\left(net_1, net_2\right) = \frac{\left|E_1 \cap E_2\right|}{\left|E_1 \cup E_2\right|}$$

Matrix of weighted Jaccard index between real data (left panels) and between simulations of our model for $w=1$ and $Q=2$ (right panels). Lender-aggressor transactions. Original networks (top panels) and statistically validated networks (bottom panels).



Bidirectional links in the original and Bonferroni networks

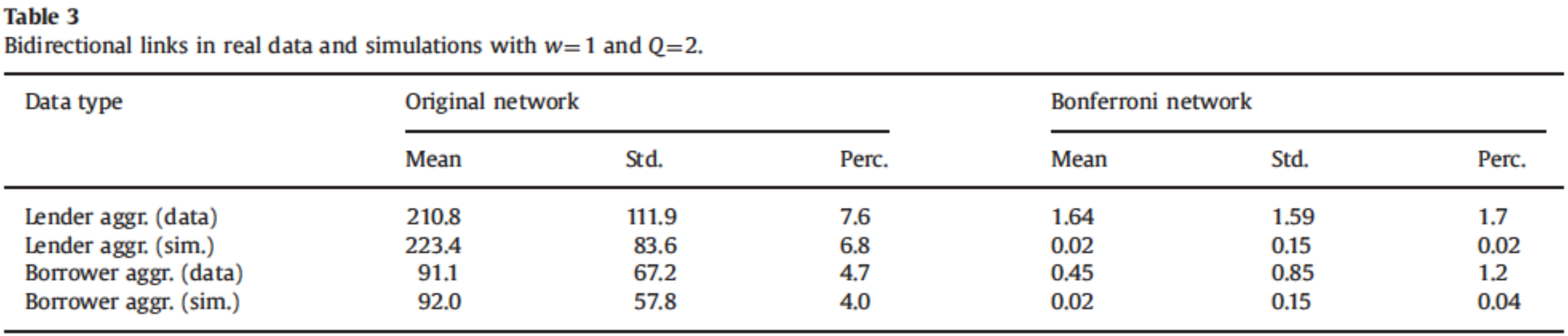


Table 3
Bidirectional links in real data and simulations with $w=1$ and $Q=2$.

Data type	Original network			Bonferroni network		
	Mean	Std.	Perc.	Mean	Std.	Perc.
Lender aggr. (data)	210.8	111.9	7.6	1.64	1.59	1.7
Lender aggr. (sim.)	223.4	83.6	6.8	0.02	0.15	0.02
Borrower aggr. (data)	91.1	67.2	4.7	0.45	0.85	1.2
Borrower aggr. (sim.)	92.0	57.8	4.0	0.02	0.15	0.04

Our model is not able to explain the percent of bidirectional links observed in Bonferroni networks

3-motifs

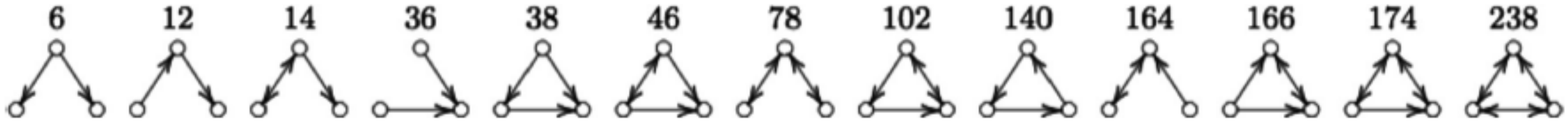


Fig. 9. The 13 isomorphic directed 3-motifs. The numeric code is the one used by the FANMOD program.

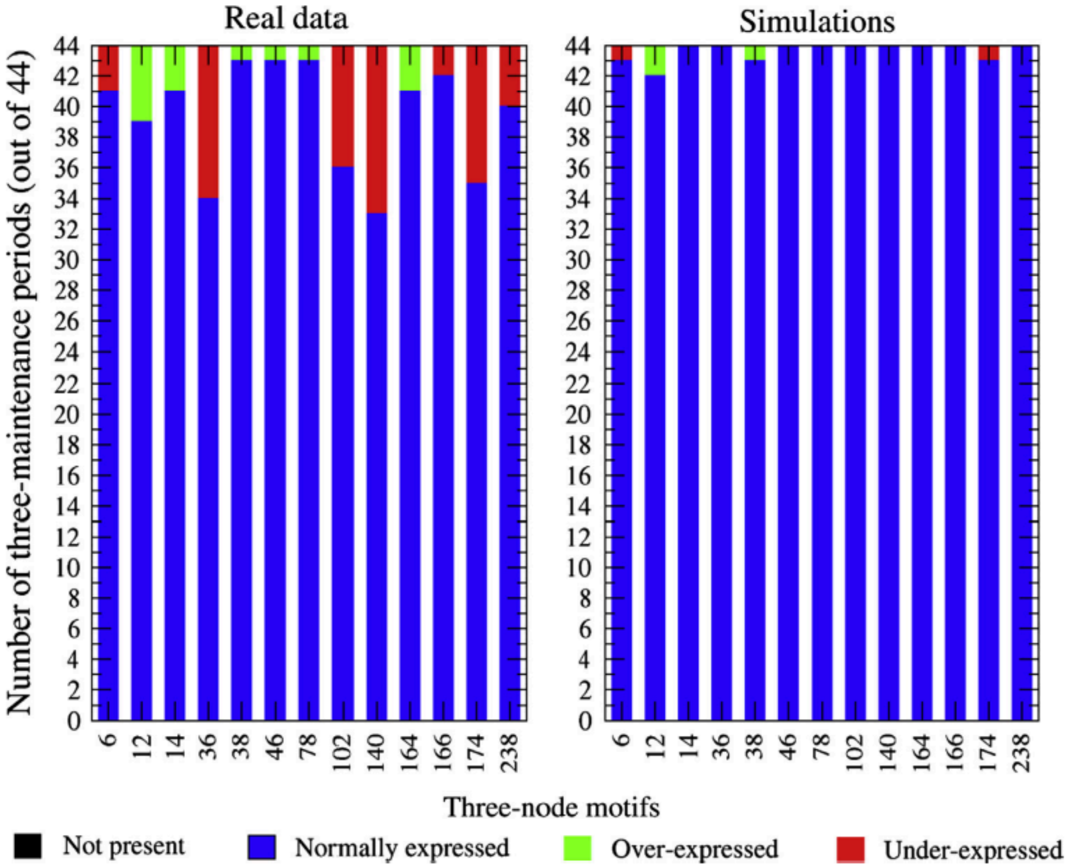


Fig. 10. Number of three-maintenance periods (out of 44) in which each three-motif type (indicated in the horizontal axis) is over-expressed, under-expressed, normally expressed, and not present in the original (left panel) networks associated with real data of *lender-aggressor* and with corresponding *lender-aggressor* transactions from simulations of the model with $w=1$ and $Q=2$ (top left). Over-expressions and under expressions are obtained by performing a multiple hypothesis test correction. The over/under expression of a three motif indicates that the corresponding p-value provided by FANMOD was smaller than $0.01/(13 \cdot 44)$, where 13 is the number of three-motif types and 44 is the number of three-maintenance periods investigated.

The over-expression or under-expression of 3-motifs in the original network of real data is not explained by the model.