“Financial Networks” Economic Links, Counterparty Risk and Network Debt

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Economic Links and Counterparty Risk

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Counterparty Risk

- Counterparty risk is an important determinant of corporate risk and therefore credit spreads.
- We describe a model of financial networks that is suitable for the construction of proxies of counterparty risk.
- With the U.S. supplier-customer network of public companies, counterparties' leverage and jump risk are significant determinants of corporate credit spreads for each supplier.
- Our findings are robust after controlling for several idiosyncratic, industry, and market factors.

Supplier-Customer (SC) networks and Counterparty risk

Risks originate and propagate in a SC network through two primary mechanisms.

1. *Trade Credit Exposure:*
   - Trade credits are extended whenever payment is not on delivery.
   - Lender takes on a risk exposure, whose magnitude depends on the size of the trade and the credit worthiness of the borrower.

2. *Future Cash Flow Risk:*
   - Strong ties along the supply-chain are valuable.
   - Arise from sharing technical knowledge, investing in specific equipment.
   - Firms are co-invested in each others' businesses.

Leverage and Jump

- The magnitude of network effects is substantial: for a given firm, an increase of one standard deviation in the leverage of its main customers leads to a widening of its credit spread of 26 basis points on average.
- This is compelling when compared to the effect of a firm's own leverage: an increase of a standard deviation in a firm's own leverage widens its credit spread by 50 basis points.
- A customer with higher leverage has on average wider spreads and, hence, a higher implied probability of default. This, in turn, reflects negatively on the supplier's prospects, and it eventually leads to a higher spread.

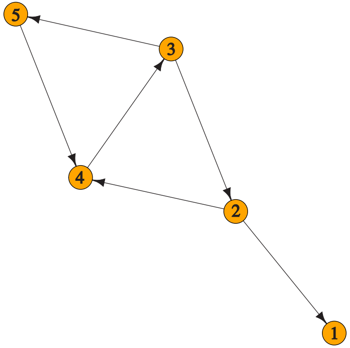
Modeling Networks

- Network Economics addresses two issues:
  - Analysis of network effects;
  - Process of network formation.
- We introduce a parametric framework for modeling network effects.
- Study the market valuation of counterparty risk.

Networks and Graphs

- Networks can be represented by graphs;
- A graph g is a pair (V, E):
  - V: set of vertices (i.e. *nodes*);
  - E: set of edges (i.e. *links*).
- A graph can be
  - *Directed*: uni-directional edges;
  - *Undirected*: bi-directional edges;
  - *Weighted*: each edge has a specific strength, quantified by a real number — its *weight*.
- We assume g has no self-loops.

Adjacency Matrix



$$
\begin{array}{cc}
 & \text{to} \\
\text{from} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$

Network Lag Operator

The adjacency matrix *acts* on vectors of *vertex characteristics.*
x: n-dimensional vector, s.t. x_i is some property of node i.

- G: adjacency matrix of g.
- The entries of Gx are *sums* of neighbors characteristics:

$$(Gx)_i = \sum_{j \in V} G_{ij} x_j = \sum_{j | i \to j} x_j \ ,$$

- G is a weighted adjacency matrix that is stochastic

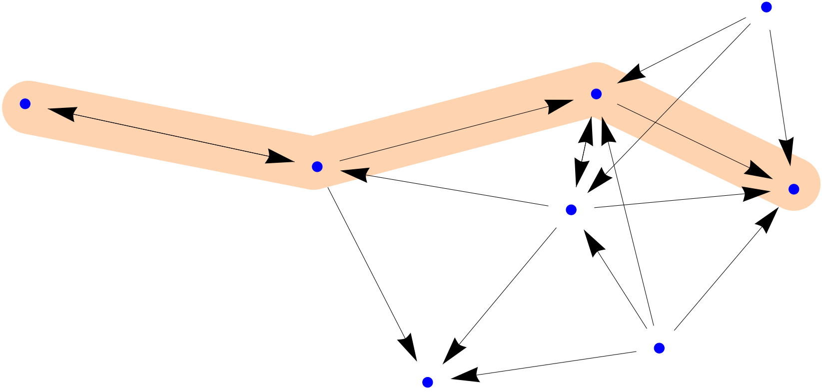
$$(Gx)_i = \sum_{j \in V} G_{ij} x_j = \sum_{j | i \to j} G_{ij} x_j$$

Network Lag Operator

$$g = \begin{tikzcd} & v_1 \arrow[dl] \arrow[dr] & \\ v_2 \arrow[dr] & & v_3 \arrow[dl] \\ & v_4 & \end{tikzcd}$$

$$Gx = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 + x_3 \\ x_4 \\ x_4 \\ 0 \end{pmatrix}$$

Walks and Powers of the Adjacency Matrix



The NARMA model

NARMA: *Network* Auto-Regressive Moving Average;

- A NARMA process of order (p, q) for y on a network g that satisfies the equation

$$y = \sum_{i=1}^{p} \alpha_i G^i y + \sum_{j=0}^{q} \beta_q G^j x + \epsilon \ ,$$

Economic Links and Counterparty Risk

- We study the network determinants of corporate credit spreads;
- Customer-supplier network:
  - *Nodes*: US public companies;
  - *Links*: Supplier $\rightarrow$ Customer (from accounting data);
- Network effects as a proxy for counterparty risk.

The Model: Network Spillovers

$$
\begin{aligned}
CS_{i,t} = \alpha + \beta\ Firm_{i,t} + \gamma\ Customers_{i,t} \\
+ \delta_1\ S\&P_t + \delta_2\ YieldCurve_t + \epsilon_{i,t}
\end{aligned}
$$

- $CS_{i,t}$ is the credit spread for of firm i at time t
- $Firm_{i,t}$ is a vector of the firm's characteristics:

$$Firm_{i,t} = \{\ lev_{i,t},\ ivol_{i,t},\ jump_{i,t}\ \}$$

- $Customers_{i,t}$ is a vector of the customers' characteristics:

$$Customers_{i,t} = \{\ (G_t \cdot lev_t)_i,\ (G_t \cdot ivol_t)_i,\ (G_t \cdot jump_t)_i\ \}$$

Data

The data is combined from several sources. The analysis is carried out on weekly data for the 2004-2009 period.

1. COMPUSTAT: customer segment files;
2. TRACE: bond transactions;
3. OptionMetrics: implied volatilities;
4. CRSP: stock prices.
5. DATASTREAM: bond characteristics, benchmark treasuries;

Credit Spreads from TRACE

1. Sample: bonds with no optionality and fixed coupon;
2. Daily Yield: volume weighted average yield from transaction data;
3. Yield Curve: linear interpolation of benchmark treasury rates from Datastream;
4. Credit spreads are computed as differences from the yield curve.

Leverage from Compustat and CRSP

For each firm i, we define firm leverage $lev_{i,t}$ as

$$\frac{Book\ Value\ of\ Debt}{Market\ Value\ of\ Equity + Book\ Value\ of\ Debt}.$$

Implied Volatility and Jump Measure from OptionMetrics

1. OptionMetrics contains the volatility surface constructed via kernel smoothing on a fixed grid of maturities and deltas for US equity option market.
2. A proxy for jump risk is the "smirk" of implied volatilities.
3. Following Yan (2010, JFE), we use near money puts and calls to estimate implied volatility

$$ivol = 0.5\left(\sigma_{i,\mathrm{put}}^{\mathrm{imp}}(-0.5) + \sigma_{i,\mathrm{call}}^{\mathrm{imp}}(0.5)\right) \ ,$$

and the slope of the volatility smile

$$jump = \sigma_{i,\mathrm{put}}^{\mathrm{imp}}(-0.5) - \sigma_{i,\mathrm{call}}^{\mathrm{imp}}(0.5) \ .$$

Economic Links from COMPUSTAT

1. COMPUSTAT Customer segment files contain the identity of principal customers;
2. Since customer's names are self-reported, matching a customer's name with a standard identifier is not straightforward. Our procedure:
   - first pass: exact matches;
   - second pass: manual match of unmatched customers.
3. We identify 4,462 firms and 21,000 links (between 2003 and 2009.)
4. After matching the firms in the supplier-customer network with the corporate bond trades in TRACE, with the bond characteristics from DataStream, and dropping missing observations, our final sample consists of 154 firms and 12,128 weekly observations.

Summary Statistics

		Mean	SD	Min	Max	Obs
All Maturities (154 Firms)						
	Credit spread	2.926	3.115	.115	29.261	12128
Implied volatility	Firm	.3618	.2283	.0856	2.3637	12128
	Customers (all)	.0606	.1255	0	2.0126	12128
	Customers	.2556	.1288	0.1072	2.0126	2694
	S&P	.1876	.0967	.0953	.6076	12128
Implied jump measure	Firm	.0090	.0431	-.6022	.8817	12128
	Customers (all)	.0009	.0078	-.2642	.2813	12128
	Customers	.0039	.0148	-.2642	.2813	2694
	S&P	.0016	.0091	-.0395	.0355	12128
Leverage	Firm	.3386	.2157	.0123	.9793	12128
	Customers (all)	.0570	.1508	0	.9992	12128
	Customers	.2440	.2275	.0008	.9992	2668
Weekly returns	S&P	.0010	.0268	-.1952	.1167	12128
Term Structure	r^{10}	4.136	.6404	2.1	5.2	12128
	slope	1.006	.9555	-.19	2.7	12128

		January 2004–December 2009						
		Classical Models				Network Spillovers		
		[1]	[2]	[3]	[4]	[5]	[6]	[7]
	(intercept)	-1.232	1.926	-1.553	1.910	-1.322	1.653	1.822
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)
Firm	lev, β_1	2.019	2.243	2.271	2.230	2.130	2.313	2.281
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	ivol, β_2	9.294	8.495	8.561	8.529	9.096	8.366	8.465
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	jump, β_3	12.450	12.977	12.958	12.853	13.027	13.543	13.367
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)
Customers	lev, γ_1					0.803	1.199	1.137
						(0.104)	(0.015)	(0.030)
	ivol, γ_2					0.671	0.256	0.337
						(0.140)	(0.540)	(0.457)
	jump, γ_3					22.547	21.344	20.630
						(0.000)	(0.001)	(0.001)
S&P	ret, $\delta_{1,1}$			4.699	3.964			3.665
				(0.000)	(0.000)			(0.000)
	ivol, $\delta_{1,2}$			2.615	0.162			-0.294
				(0.001)	(0.819)			(0.693)
	jump, $\delta_{1,3}$			2.471	-1.430			-0.848
				(0.556)	(0.547)			(0.736)
Yield Curve	r^{10}, $\delta_{2,1}$		-0.681		-0.684		-0.641	-0.674
			(0.000)		(0.000)		(0.000)	(0.000)
	slope, $\delta_{2,2}$		-0.128		-0.139		-0.126	-0.129
			(0.012)		(0.003)		(0.015)	(0.007)
Degrees of freedom		12124	12122	12121	12119	11175	11173	11170
R^2 (adjusted)		0.683	0.693	0.689	0.694	0.690	0.698	0.699

Cross-Industry Effects

A common critique:

This model does not capture network effects but cross-industry spillovers.

- Averaging over customers' characteristic builds proxies for whole industrial sectors
- We conduct a robustness check that rejects this hypothesis.

Cross-Industry Effects

- Control variables for industry and cross-industries;
- Customers $\Longrightarrow$ neighboring industries
- The model with industry effects is

$$y = \underbrace{\beta\, Firm + \gamma\,(G \cdot Firm)}_{\substack{\text{Firm and}\\ \text{Customers}\\ \text{effects}}} + \underbrace{\delta\,(S\&P,\, YieldCurve)}_{\substack{\text{Market}\\ \text{effects}}} + \underbrace{\eta\, Ind + \phi\,(G \cdot Ind)}_{\substack{\text{Industry and}\\ \text{Cross-industry}\\ \text{effects}}}$$

		No Industries	Industry Portfolios				
		0	12	17	30	38	48
	(intercept)	1.822	1.545	1.357	1.526	1.916	1.839
		(0.000)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)
Firm	lev, β_1	2.281	2.221	2.236	2.239	2.275	2.251
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	ivol, β_2	8.465	8.486	8.512	8.503	8.471	8.476
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	jump, β_3	13.367	13.493	13.527	13.509	13.342	13.396
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Customers	lev, γ_1	1.137	1.205	1.163	1.129	1.206	1.259
		(0.030)	(0.021)	(0.030)	(0.037)	(0.022)	(0.013)
	ivol, γ_2	0.337	0.097	0.237	0.338	0.142	-0.049
		(0.457)	(0.837)	(0.643)	(0.510)	(0.785)	(0.914)
	jump, γ_3	20.630	20.899	21.601	21.331	20.556	19.536
		(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
Industry	ret, η_1		0.003	0.002	0.008	-0.014	-0.003
			(0.672)	(0.840)	(0.236)	(0.012)	(0.579)
	vol, η_2		-0.009	-0.010	-0.006	0.000	-0.001
			(0.000)	(0.000)	(0.000)	(0.846)	(0.440)
Cross-Industry	ret, ϕ_1		-0.048	-0.039	-0.035	-0.009	-0.022
			(0.004)	(0.001)	(0.001)	(0.239)	(0.075)
	vol, ϕ_2		0.017	0.002	-0.001	0.002	0.005
			(0.000)	(0.648)	(0.792)	(0.133)	(0.032)
Degrees of freedom		11170	11166	11166	11166	11166	11166
R^2 (adjusted)		(0.699)	(0.701)	(0.001)	(0.701)	(0.700)	(0.700)

Conclusions

- The main objective of this paper is to evaluate the market assessment of counterparty risk in supplier-customer relationships;
- We study the network determinants of corporate credit spreads and use network effects as an instrument for counterparty risk.
- Counterparty risk is an economically and statistically significant determinants of credit spreads.

Network Debt

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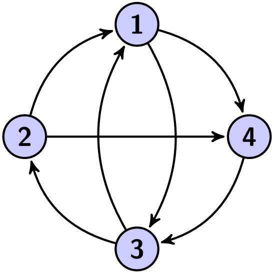
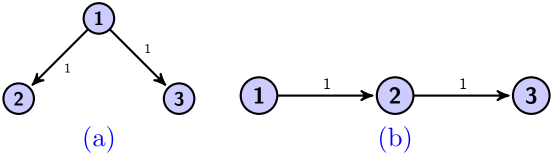
The problem



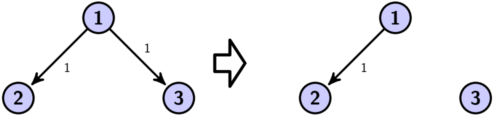
- A financial liability network of n economic agents.
- Described by a weighted directed graph. Vertices are agents. A weighted arrow from i to j denotes agent i lending to agent j and how much.
- Default caused by exogenous shock, e.g. unanticipated liquidity demand external to network.

- The network (a), agents 2 and 3 hold one unit of asset each from agent 1—e.g. two entrepreneurs borrowing from a principal.

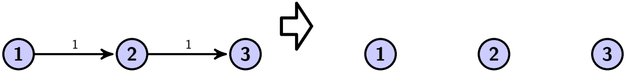
- In (b), one unit of asset is passed from agent 1 to 2, then from 2 to 3—e.g. agent 2 can be a frictionless intermediary between an entrepreneur and a principal.



If agent 3 defaults in network (a), he is removed from the network. Agents 1 and 2 remain.



In network (b), agent 3 hold the only asset internal to the network, his default causes the entire network to collapse.

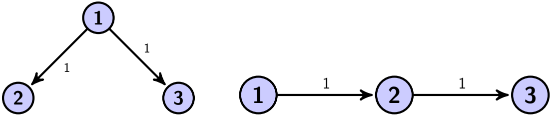


- The default of an agent propagates through the network.
- A default is severe if it propagates through a large part of the network, e.g. agent 3's default is a more severe event for network (b) than (a).
- Given a network of arbitrary liability structure, the default of different agents have different severity.

How to quantitatively compare the severity of each agent's potential default in a given network?

What is wrong with simple accounting?

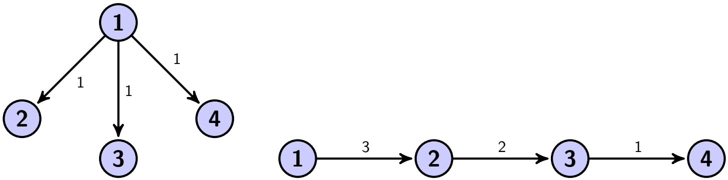
- Any proposed answer must be able to distinguish between different liability structures. Simply looking at each agent's balance sheet does not achieve this. The same amount of inter-agent liability can have different default-propagation effects across networks.



In both networks (a) and (b), agent 3 has liability of 1 unit. In (a), his default removes only him from the network while in b, his default causes the entire network to collapse. These two events are not of the same severity.

What is wrong with simple accounting?

The liability profile of entire network also does not capture the network structure.



These two networks have identical agent balance sheets: $\{-3, 1, 1, 1\}$. The network structures are very different and therefore so are the propagation effects.

A proposed solution

- We would like a notion that focuses on the severity of an agent's default in terms of implications for the network—*how much of the network does the agent take with him when he defaults?*

- We expect owing to a heavily in-debt creditor means more severe implications of default than owing to a debt-free creditor. A default on a heavily in-debt creditor is more likely to propagate.

- We will start with an intuitive expression and drive toward a precise graph-theoretical measure of severity.

A proposed solution

Instead of a simple sum ($d_i = 1$), one can envision attaching a weight d_i to each agent i, for $i = 1, \cdots, n$ such that each d_i is a weighted sum of agent i's liabilities $\{a_{ij},\ j = 1, \cdots, n\}$

$$d_i = \sum_{j=1}^{n} a_{ij} d_j,\ 1 \leq i \leq n.$$

This immediately captures:

- Agent i having higher debt increase his own weight d_i.
- For the same amount owed, d_i increases with d_{-i}, the weight of his creditors.

As an agent increases his own liability, both his own weight and the weights of those who borrow from him increase.

A definition: network debt distribution

The equation characterizes $d = (d_i)$ as an eigenvector of the adjacency matrix $A = [a_{ij}]$, $Ad = d$, corresponding to eigenvalue 1. Allowing for a scaling factor λ gives $Ad = \lambda d$. The entry $a_{ij} \geq 0$ is the amount i borrows from j.

Definition

Given a (strongly connected) financial network of n agents with non-negative edge weights $\{a_{ij},\ 1 \leq i, j \leq n\}$, let d be an eigenvector of $A = [a_{ij}]$ corresponding to the positive eigenvalue λ with maximum modulus. Normalize d so that $\sum_1^n d_i = 1$. The vector d is the *network debt distribution* of the network. The i-th entry d_i of d is the *(relative) network debt of agent i*.

(By *modulus* of a complex number we mean its length as a two-dimensional vector.)

Comments on the definition

- Allowing for the scaling factor λ loses no information. The normalization $\sum_1^n d_i = 1$ means we are interested in the ratios $\frac{d_i}{d_j}$, $i \neq j$.
- Allowing for λ also ensures the existence of d. $Ad = \lambda d$ always has solutions while $Ad = d$ need not. An $n \times n$ matrix (even with non-negative entries) have n complex eigenvalues in general. 1 need not be an eigenvalue and, even if it is, the corresponding eigenvector need not be unique and positive.
- Choosing λ to be positive with maximum modulus ensures there is a d with positive entries that is unique up to scale (Perron-Frobenius theorem). So d is really a distribution and well-defined.

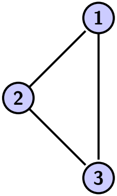
Graph-theoretic perspective

In algebraic graph theory, λ and d are called the *maximal eigenvalue* and *maximal eigenvector* of the graph. They capture certain properties of the underlying graph. For example,

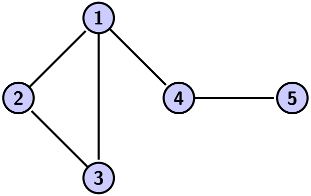
- The *maximum clique problem*: what the largest group of agents who have liability relationship to each other?/what is the largest group of economists who are co-authors with each other? *Answer:* $\lambda + 1$ is an upper bound on the maximum clique size of the graph.

- The *maximum clique problem* cont'd

  For example, the graph (k)



(k)



(l)

  form a clique, and therefore has maximum clique size $3 = 2 + 1$—its adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

  has maximal eigenvalue 2. In graph (l), vertices $\{1, 2, 3\}$ form a maximum clique.

- Google's PageRank algorithm determines the ranking of web-pages by computing the maximal eigenvector of the internet. In the internet graph, an arrow from i to j represents site i linking to site j. In the internet of n sites, the maximal eigenvector (v_i) is an n-dimensional vector with positive entries. The v_i's provide a ranking of the sites. A heavily linked site, e.g. `www.themoscowtimes.com`, receives high ranking.

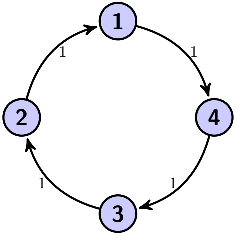
Is this a valid notion for financial networks?

- The Google example suggests that, for our context, a highly leveraged agent should have high weights in the distribution specified by d—an arrow from i to j means agent i lending to agent j. However, does it reflect the severity of default implications as intended?

- How does it behave when the liability structure is perturbed, i.e. when an agent borrows more or reduces her debt?

We check it against some stylized examples next.

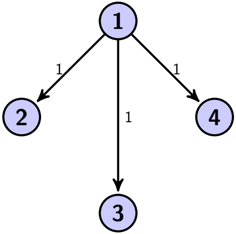
Ring network



$$A=[a_{ij}]=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- One unit of asset circulates among 4 agents.
- The eigenvalues of A are $\{1, -1, \sqrt{-1}, -\sqrt{-1}\}$ with maximal eigenvalue 1.
- Its network debt distribution is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.
- The distribution reflects that agents in this network are homogeneous. The default of any agent have the same severity. They all causes the network to collapse.

Star network

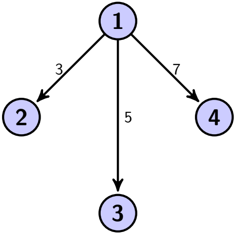


$$A=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Agents 2, 3, and 4 holds one unit each from agent 1.
- The eigenvalues of A are $\{0, 0, 0, 0\}$.
- The network debt distribution is $(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
- The default of each of the borrowers (agents 2, 3, or 4) destroys $\frac{1}{3}$ of the network.

Star network

- Changing the liabilities (edge weights) of the star network while keeping the star structure does not change the network debt distribution because the implication of default is the same.
- For example, a star network with different amounts borrowed:



$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 \end{bmatrix}$$

- The network debt distribution is the same, $(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Line network

$$1 \xrightarrow{1} 2 \xrightarrow{1} 3 \xrightarrow{1} 4$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

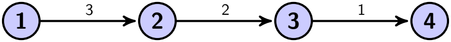
- One unit of asset is passed successively from agents 1 to 4. Agent 4 holds the only asset in this network.
- The eigenvalue of A are $\{0, 0, 0, 0\}$. The network debt distribution is $(0, 0, 0, 1)$.
- The entire network is dependent upon agent 4 to honor his debt. If 4 defaults, it propagates through the entire network.
- By our environment assumptions, the default of 3 necessarily means 4 have defaulted. Similarly for the default of 4. Any default must originate from 4 in this configuration.

Theorem

If agent i has zero liabilities, then his network debt is 0

Since $d_i = \lambda \sum_{j=1}^{n} d_j a_{ij}$, when $a_{ij} = 0$ for all j, $d_i = 0$.

The converse, however, is not true.



This line network has network debt distribution $(0, 0, 0, 1)$. The distribution is concentrated on agent 4, who is again crucial. Any default in this network must originate from that of agent 4.

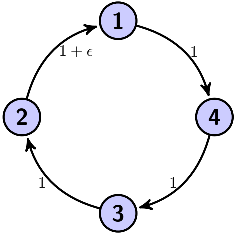
Theorem

In a (strongly connected) network of financial liabilities, suppose agent i increases its liabilities. Let $\mathbf{d}$ and $\mathbf{d}'$ be the network debt distribution before and after the increase in agent i's liability profile respectively, then

$$\frac{d_i'}{d_i} > \frac{d_j'}{d_j}, \ \ \forall j \neq i.$$

In other words, if two networks N and N' are identical except that $a_{ij}' \geq a_{ij}$ for every j, agent i must have the most network debt increase across the networks, in relative terms.

A network is *strongly connected* if for any pair of agents (i, j), there is a directed path from i to j.



$$A=\begin{bmatrix} 0 & 1+\epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

For example, in the ring network, if 1 is liable to 2 for ϵ additional units of debt, network debt distribution is, up to a normalizing factor,

$$d = (\lambda^3, 1, \lambda, \lambda^2), \quad \lambda = (1+\epsilon)^{\frac{1}{4}} > 1.$$

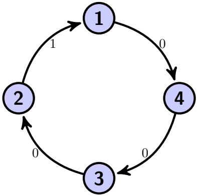
λ is the maximal eigenvalue.

An explicit calculation for λ: the eigenvalues of the adjacency matrix are solutions to the equation

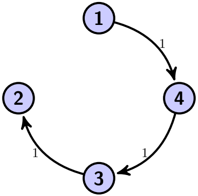
$$\det \begin{bmatrix} \lambda & -(1+\epsilon) & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -1 & 0 & 0 & \lambda \end{bmatrix} = 0.$$

i.e. $\lambda^4 = 1 + \epsilon$. So the eigenvalues are (counting algebraic multiplicity) $\{(1+\epsilon)^{\frac{1}{4}}, (1+\epsilon)^{\frac{1}{4}}, (1+\epsilon)^{\frac{1}{4}}, (1+\epsilon)^{\frac{1}{4}}\}$. In particular, the maximal eigenvalue is $(1+\epsilon)^{\frac{1}{4}}$. Eigenvalue and eigenvector calculations can be done fast numerically.

- In the above example, as ϵ gets large, this distribution approaches $(1, 0, 0, 0)$ which is the network debt distribution of the limit network (w).
- As ϵ tends to -1, the network configuration approaches the line network (x) and network debt distribution behaves correspondingly, converging to $(0, 1, 0, 0)$.



(w) $\epsilon \to \infty$
Weights in relative terms



(x) $\epsilon \to -1^+$

In other words, network debt changes smoothly with respect to the liability structure.

Theorem

Let Δr be a non-negative vector. Consider the perturbations of the network where are agent i increase his liabilities by $\eta \cdot \Delta r$ with network debt $d(\eta)$, normalized so that $d_i(\eta) = 1$. Then $d_j(\eta)$ is a differentiable function of η, for $j \neq i$.

Δr is vector of the additional amounts borrowed by agent i.

Theorem

In a (strongly connected) network of financial liabilities, suppose agent i increases its liabilities. Let d and d' be the network debt distribution before and after the increase in agent i's liability profile respectively. In the same scenario and same notion as above, if $d'_i = 0$, then d' and d are the same.

If, after increasing its liabilities, agent i is found to have no relative network debt, then it already has zero network debt prior to increase. Furthermore, the entire network debt distribution remains the same. Consider, for example, perturbing a line network.

$$1 \xrightarrow{3} 2 \xrightarrow{2+\epsilon} 3 \xrightarrow{1} 4$$

Consider, for example, perturbing a line network.

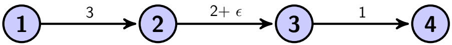
$$1 \xrightarrow{3} 2 \xrightarrow{2+\,\epsilon} 3 \xrightarrow{1} 4$$

The network debt distribution for $\epsilon = 0$ is $(0, 0, 0, 1)$. Because any default must originate from 4. This remains the case when 3 borrows ϵ more from 2. Therefore the network debt distribution is the same.

Theorem

In the same scenario and same notion as above, if the additional liabilities incurred by agent i is to agents j such that $d_j = 0$, then d' and d are the same.

If an agent borrows more from a zero network debt creditor, the network debt, for the entire network, is unaffected. This is consistent with what we intend to capture but the general proof is not entirely trivial.



$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2+\epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- For example, in a line network, the intermediary agent 2 has network debt 0.
- If agent 3 borrows ϵ more from agent 2, the network debt distribution remains the same: $(0, 0, 0, 1)$.
- This reflects that additional borrowing by agent 3 does not change potential default implications within the network.

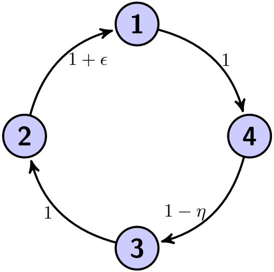
Theorem

Consider a strongly connected network where agent i increases his liabilities and agent j decreases hers. Then, for any $k \notin \{i, j\}$:
(i) If $\lambda \leq \lambda'$, then $\frac{d'_k}{d_k} \leq \frac{d'_i}{d_i}$.
(ii) If $\lambda \geq \lambda'$, then $\frac{d'_k}{d_k} \geq \frac{d'_j}{d_j}$.
(iii) If $\lambda = \lambda'$, then $\frac{d'_j}{d_j} \leq \frac{d'_k}{d_k} \leq \frac{d'_i}{d_i}$.

In particular, when the maximal eigenvalue is unaffected by perturbation (case (iii)), the conclusion is a natural one: in terms of relative increase,

`network debt of` j $\leq$ `network debt of` $k \notin \{i, j\}$ $\leq$ `network debt of` i.

(Two different liability structures can lead to same λ, e.g. star and line networks.)



$$A = \begin{bmatrix} 0 & 1+\epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-\eta \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- For example, if ϵ is large relative to η in the above perturbation of the ring network, then $\lambda' > \lambda$. This is case (i) and the conclusion of the theorem puts focus on the agent who borrowed more.
- Conversely, if ϵ is small relative to η, this results in case (ii) and the theorem's conclusion focuses on the agent who reduced her liabilities.

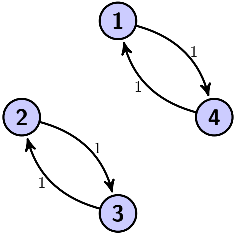
The maximal eigenvalue of the previous network solves the equation

$$\det \begin{bmatrix} \lambda' & -(1+\epsilon) & 0 & 0 \\ 0 & \lambda' & -1 & 0 \\ 0 & 0 & \lambda' & -(1-\eta) \\ -1 & 0 & 0 & \lambda' \end{bmatrix} = (\lambda')^4 - (1+\epsilon)(1-\eta) = 0$$

- If $(1+\epsilon)(1-\eta) > 1$, then $\lambda' > 1 = \lambda$, we are in case (i).
- If $(1+\epsilon)(1-\eta) < 1$, then $\lambda' < 1 = \lambda$, we are in case (ii).

d does not see components

A limitation of this notion is that it does not see components. For example, this network also has network debt distribution $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, same as the ring network. It has two connected components, while the ring network has one.



One default here destroys half (one component) of the network, instead of the entire network, although agents are still homogeneous.

Therefore, to fully exploit this notion, it should be accompanied by cluster analysis that separate the network into components or, more generally, weakly connected components.

Future Research/Applications

- d is immediately computable—financial networks are small compared to the internet. This is a tool that can aid regulators in identifying potential vulnerabilities in a financial network.

- The above is a snap-shot analysis, causes for change in configuration are taken as exogenous. One can incorporate the evolution of network debt distribution in a economic model, linking network debt distribution to market equilibrium.