# Multifractal Random Walk and its connection with exo, endo market shocks identification


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# Basic "Stylized facts" of asset returns

1. Heavy tails of asset returns distribution
2. Aggregational gaussianity
3. Absence of autocorrelation in asset returns
4. Long memory in volatility
5. Volatility clustering
6. Multifractal properties of asset price time series
7. Time reversal asymmetry and leverage effect
8. Extreme events (Bubbles and crushes)

- Geometrical Brownian Motion - too rough approximation of the real processes
- Fractal Geometrical Brownian Motion has memory not only in volatility but also in returns
- AR, ARCH, GARCH models (in total more than 50) are very specialized:
  - GARCH reproduce heavy tails of PDF
  - FIGARCH reproduce slow declining volatility's ACF
  - ERACH reproduce leverage effect
- Multiplicative Cascades Model is difficult for interpretation in applications
- Thus the problem of creating a model, which takes into account basic stylized facts, is still actual

# Definition of the multifractality

**Geometrical definition**

Fractal is a structure consisting of substructures, each of which is geometrically similar to hole structure

**Abstract definition**

Process that preserves its statistical properties under arbitrary affine transformations

Multifractal or monofractal nature of the random process can be determined from the analysis of its absolute moments:

$$M_q\left(l\right) = \langle |\delta_l X\left(t\right)|^q \rangle = \langle |X(t+l) - X(t)|^q \rangle \tag{1}$$

which for multifractal process can be represented in the form

$$M_q\left(l\right) = K_q l^{\zeta_q} \tag{2}$$

where $\zeta_q$ - non-linear function of q

# Multifractal Random Walk model

MRW model was the first model with stationary increments and clear dependency with respect to time. The authors proposed a continuous random process:

$$X(t) = \lim_{\Delta t \to 0} X_{\Delta t}(t) \tag{3}$$

as a limit of discrete random process

$$X_{\Delta t}(t) = \sum_{k=1}^{t/\Delta t} \delta X_{\Delta t}[\Delta t \; k] = \sum_{k=1}^{t/\Delta t} \xi_{\Delta t}[k] \, e^{\omega_{\Delta t}[k]} \tag{4}$$

where

- $\xi_{\Delta t}[k]$ - Gaussian white nose with zero mean and variance $\sigma^2 \Delta t$
- $\omega_{\Delta t}[k]$ - independent from $\xi[k]$ correlated Gaussian random process
- $\Delta t$ - discretization step of the random process

In financial application $X(t)$ can be employed as the process for logarithm of price

In the MRW model $\omega_{\Delta t}[k]$ is employed as stochastic log-volatility with zero mean and covariance function decreasing logarithmically:

$$\text{Cov}\left[\omega_{\Delta t}[k_1], \omega_{\Delta t}[k_2]\right] = \lambda^2 \ln \rho_{\Delta t}\left[|k_1 - k_2|\right] \tag{5}$$

where

$$\rho_{\Delta t}[k] = \begin{cases} \frac{L}{(|k|+1)\,\Delta t}, & \text{if } |k| \le \frac{L}{\Delta t} - 1 \\ 1, & \text{if } |k| > \frac{L}{\Delta t} - 1 \end{cases}$$

Here manifest itself the first disadvantage of MRW model

$$\lim_{\Delta t \to 0} \langle \omega_{\Delta t}[k]^2 \rangle = \lim_{\Delta t \to 0} \lambda^2 \ln \frac{L}{\Delta t} = \infty \tag{6}$$

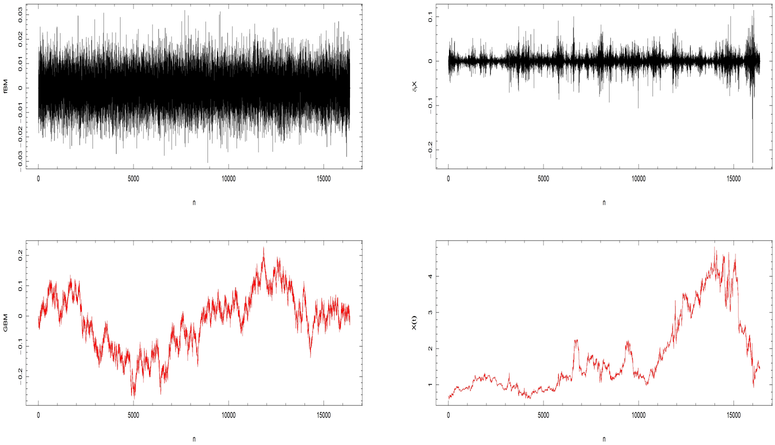
On scales $l \le L$ MRW model has strict multifractal properties and parabolic spectrum:

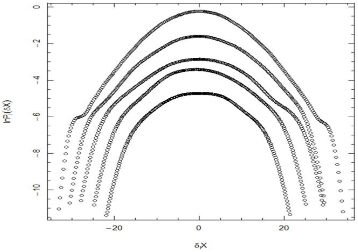
$$\zeta_q = \left(\frac{1}{2} + \lambda^2\right) q - \frac{\lambda^2}{2} q^2 \tag{7}$$

On scales $l > L$ MRW model has monofractal properties and linear spectrum $\zeta_q = q/2$

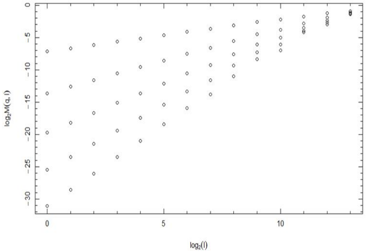
# Multifractal Random Walk model
## Simulation

Fractal Brownian Motion with (left) versus Multifractal Random Walk (right)

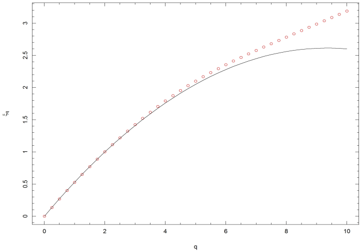




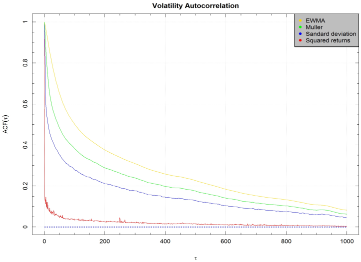
lnP(δX)
δX
−20
0
20

log2M(q, l)
log2(l)

ξq
q

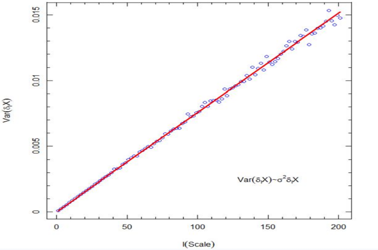
Volatility Autocorrelation
EWMA
Muller
Sandard deviation
Squared returns
ACF(τ)
τ

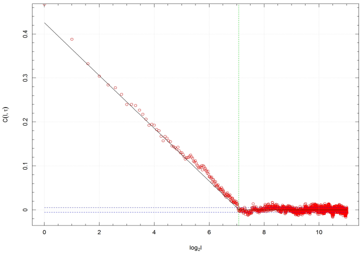
Parameter $\sigma^2$ can be estimated by means of fitting dependency between MRW increment's scales and their variance (left fig.):

$$\text{Var}\left[\delta_l X_{\Delta t}\left[k\right]\right] = \sigma^2 l \tag{8}$$

Remaining parameters $\lambda^2$ and $L$ can be obtained by means of analysis of the magnitude correlation function (right fig.):

$$\hat{C}_\tau\left(l\right) = \langle\left|\delta_\tau X\left[k+l\right]\right|, \left|\delta_\tau X\left[k\right]\right|\rangle, \qquad C_\tau\left(l\right) \sim -\lambda^2 \ln\left(\frac{l}{L}\right) \tag{9}$$



Skewed MRW is obtained from the MRW model by correction of log-volatility process:

$$\delta X_{\Delta t}\left[k\right] = \xi_{\Delta t}\left[k\right] e^{\omega_{\Delta t}\left[k\right]}, \qquad \tilde{\omega}_{\Delta t}\left[k\right] = \omega_{\Delta t}\left[k\right] - \sum_{i=1}^{i<k} K\left(i,j\right)\,\xi_{\Delta t}\left[k\right] \qquad (10)$$

where $K\left(i,j\right)$ - power-law kernel describing how the sign of the returns at time i affect at the log-volatility at time j

$$K\left(i,j\right) = \frac{K_0}{\left(j-i\right)^{\alpha}\ \Delta t^{\beta}}, \quad j > i \qquad (11)$$

Ratio $\frac{K_0}{\Delta t^{\beta}}$ can be estimated by means of empirical and asymptotic analytical leverage functions:

$$\hat{L}\left(\tau\right) = \frac{\langle \delta X_{\Delta t}\left[i\right], \delta X_{\Delta t}\left[i+\tau\right]\rangle}{\langle \delta X_{\Delta t}\left[i\right]^2\rangle^{3/2}},\ L\left(i,j\right) = -2\left(\frac{L}{\Delta t}\right)^{\frac{3\lambda^2}{2}} \frac{K\left(i,j\right)}{\left|i-j\right|^{2\lambda^2}} \qquad (12)$$

# MRW as a possible bridge to market shock origins

"Are large market events caused by easily identifiable exogenous shocks such as major news events, or can they occur endogenously, without apparent external cause, as inherent property of market itself?"

This question was posted by Didier Sornette, Yannick Malevergne and Jean-Francois Muzy

Some thoughts about market shocks:

- Market shocks occur on most of the world's stock markets: October 1987, the Hong Kong crash, the Russian Default in August 1998...
- There is no doubt that some of the large market shocks are results from really bad news that moves stock market prices and creates strong bursts of volatility (like the event of September 11, 2001 and the coup against Gorbachev on August 19, 1991)
- But not all crashes seem to be cause by exogenous forces. Several researchers have looked for more fundamental origins and have proposed that a crash may be the climax of an endogenous instabilities
- Even more difficult is classification of the volatility bursts: endogenous versus exogenous

# MRW as a possible bridge to market shock origins

**Exogenous shock**

Response of the system to a single piece of very bad news that is sufficient by itself to move the market significantly

The authors consider situation, when system experiences an external shock with amplitude $\omega_0$ at $t = 0$:

$$\omega(t) = \mu + \int_{-\infty}^{t} \left[\omega_0 \times \delta(\tau) + \eta(\tau)\right] K(t-\tau)\, d\tau \tag{13}$$

The expected volatility conditional on this shock has form:

$$E_{exo}\left[\sigma^2(t)\,|\omega_0\right] = E_{exo}\left[e^{2\omega(t)}|\omega_0\right] = \bar{\sigma}^2(t)\, e^{2\omega_0 K_0 \sqrt{\frac{\lambda^2 L}{t}}} \tag{14}$$

For large enough t the volatility relaxes to its unconditional average $\bar{\sigma}^2(t) = \sigma^2 \Delta t$. So, that the excess volatility due to the external shock decays to zero as:

$$E_{exo}\left[\sigma^2(t)\,|\omega_0\right] - \bar{\sigma}^2(t) \sim \frac{1}{\sqrt{t}} \tag{15}$$

# MRW as a possible bridge to market shock origins

## Endogenous shock

Response of the system to the cumulative effect of many small pieces of bad news, each one looking relatively benign on its oven but together adding up, due to long memory of the volatility, to create large endogenous shock

- In these case authors consider the natural evolution of the system without any large shock, which nevertheless exhibit a large volatility burst $\omega_0$ at $t = 0$
- Large endogenous shock requires a special set of realization of the small pieces of news $\eta(t)$

Thus, to qualify the response in that case authors evaluate:

$$E_{endo}\left[\sigma^2(t)\,|\omega_0\right] = E_{endo}\left[e^{2\omega(t)}|\omega_0\right] = \\ = \exp\left(2E\left[\omega(t)\,|\omega_0\right] + 2Var\left[\omega(t)\,|\omega_0\right]\right) \quad (16)$$

# MRW as a possible bridge to market shock origins

After all calculations volatility response conditional to endogenous shock has form

$$E_{endo}\left[\sigma^2(t)\,|\omega_0\right] = \bar{\sigma}^2(t)\left(\frac{L}{t}\right)^{\alpha(s)+\beta(t)} \tag{17}$$

where $\alpha(s)$ - conditional volatility response exponent

$$\alpha(s) = \frac{2s}{\ln\left(\frac{Le^{3/2}}{\Delta t}\right)}, \quad \beta(t) = 2\lambda^2\frac{\ln(t/\Delta t)}{\ln\left(Le^{3/2}/\Delta t\right)}, \quad \omega_0 - \mu = s + C(0) \tag{18}$$

Within the range $\Delta t < t \leq \Delta t e^{\frac{|s|}{\lambda^2}}$ $\beta(t) \ll \alpha(s)$ and equation (17) leads to a power-law behaviour:

$$E_{endo}\left[\sigma^2(t)\,|\omega_0\right] \sim t^{-\alpha(s)} \tag{19}$$

For determination of the source of the endogenous shock the authors consider the process

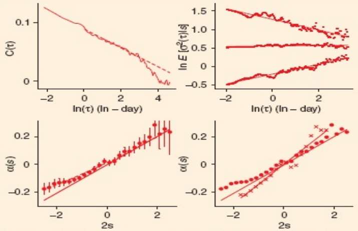
$$W(t) = \int_{-\infty}^{t}\eta(\tau)\,d\tau \tag{20}$$

They conclude that expected path of information flow prior to the endogenous shock (for $t < 0$) grows like $\Delta t/\sqrt{-t}$

# MRW as a possible bridge to market shock origins

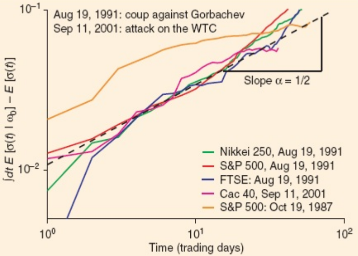
Figures from Didier Sornette, Yannick Malevergne and Jean-Francois Muzy: What causes crashes?

**2. Conditional volatility response exponent $\alpha(s)$ for S&P 100 intra-daily time series**




Conditional volatility response exponent $\alpha(s)$ for S&P 100 five-minute intra-daily time series (from April 8, 1997 to December 24, 2001) as a function of the endogenous shock amplitude. Top-left panel: 40-minute log-volatility covariance $C_{40}(\tau)$ as a function of the logarithm of the lag $\tau$. The MRW theoretical curve with $\lambda^2 = 0.018$ and $T = 1$ year (dashed line) provides an excellent fit of the data up to lags of one month. Top-right panel: conditional volatility response $\ln(E_{endo}[\sigma^2(t)|s])$ as a function of $\ln(t)$ for three shocks $s = -1, 0, 1$. Bottom-left panel: estimated exponent $\alpha(s)$ for $\Delta t = 40$ minutes (•) as a function of $s$. The solid line is the prediction given by equation (22). The dashed line corresponds to the prediction of the MRW obtained by averaging more than 500 Monte Carlo MRW realisations. Notice that this numerically generated MRW prediction fits more accurately the S&P 100 curve for negative values corresponding to volatilities smaller than average. This is because it accounts for the strong fluctuations associated with the smallest volatility realisations (since volatilites are calculated by summing squared returns, small volatility values are poorly estimated (see text)). The error bars correspond to 95% confidence intervals estimated from Monte Carlo trials of MRW processes. Bottom-right panel: $\alpha(s)$ is compared for $\Delta t = 40$ minutes (•) and $\Delta t = 1$ day (×) with the MRW predictions given by equation (22) shown by the solid lines

**1. Cumulative excess volatility at scale $\Delta t = 1$ day**




Cumulative excess volatility at scale $\Delta t = 1$ day, that is, integral over time of $E_{exo}[\sigma^2(t)|\omega_0] - \sigma^2(t)$, due to the volatility shock induced by the coup against President Gorbachev observed in three UK, Japanese and US indexes and the shock induced by the attack of September 11, 2001 against the World Trade Center. The dashed line is the theoretical prediction obtained by integrating (1), which gives a $\propto \sqrt{t}$ time-dependence. The cumulative excess volatility following the crash of October 1987 is shown as an orange line. Notice that the slope of the non-constant curve for the October 1987 crash is very different from the value 1/2 expected and observed for exogenous shocks. This crash and the resulting volatility relaxation can be interpreted as an endogenous event