Equilibrium on the Interest Rate Market Analysis

RNDr. Eva Kvasničková

PhD student of Econometrics and Empirical Economics
Faculty of Economics
Tor Vergata University in Rome, Italy

http://www.economia.uniroma2.it/phd/econometricsempiricaleconomics/

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- Interest Rate Models
- Convexity Adjustment
- Arbitrage Analysis

1 Interest Rate Models

- Equilibrium Models

- built on assumptions how the economy works
- aim to achieve a balance between the supply of bonds and other securities and demand for these by investors
- it may happen that the initial set of prices is different from observed market prices
- the initial term structure is an output from the model

- No-Arbitrage Models

- the initial term structure is an input to the model
- future price evolves in a way which is consistent with initial price structure and which is arbitrage free
- main advantage: designed to be exactly consistent with today's term structure

- Equilibrium Models

- assume particular properties of economic variables and derive a process for the short-rate *r*
- explore what the process implies for the prices of particular derivatives

One- Factor Models

The process for the short-rate *r* involves only one source of uncertainty.

It is usually described by an Itô process of the form:

$$dr = m(r_t)dt + s(r_t)dW_t$$

W : standard Brownian motion under the real world measure

m (instantaneous drift) and ***s*** (instantaneous standard deviation) are adapted processes, functions of *r*, independent of time itself

It implies that all rates move the same direction, but not all move by the same amount.

We will describe more in detail ***Vašíček*** and ***Cox-Ingersoll, and Ross (CIR)*** models.

Vašíček Model

- the instantaneous spot rate evolves as an Ornstein-Uhlenbeck process with constant coefficients:

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

a*, *b*, *σ are positive constants,

b represents the risk-neutral long-term mean risk-free rate;

a represents the rate at which *r* reverts back to this long-term mean;

σ represents the local volatility of short-term interests rate

Key feature: ***mean reversion*** - the interest rates appear to be pulled back to some long-run average level over time

Minuses: negative values of short-term interest rate *r,* independence of volatility and interest rate *r* (many empirical evidences suggest that the volatility of *r* is not constant, but increasing function of *r*)

Cox, Ingersoll, and Ross Model

- an alternative one-factor model for the risk-free rate of interest, where rates are always nonnegative
- the risk-neutral process in their model is:

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}\,dW_t$$

a, ***b***, ***σ*** are positive constants

- *r* cannot reach negative values
- mean-reverting drift
- with increasing short-term interest rate its standard deviation increases as well

• No-Arbitrage Models

- main disadvantage of the equilibrium models: they do not automatically fit today's term structure

- no-arbitrage models: designed to be exact-reflecting today's term structure

- one way of introducing such models is specifying Markov model for the short rate where the drift is a function of time

- many models developed in this way are natural extensions of the equilibrium models described earlier.

We will describe more in detail **Ho-Lee** and **Hull-White** models.

Ho and Lee Model

- Ho and Lee considered the following model for the risk-free rate:

$$dr_t = \theta_t dt + \sigma dW_t$$

σ is the constant instantaneous standard deviation of the short rate

θ is time dependent function chosen to ensure that the model fits the initial term structure

- Advantage: it is a Markov tractable model, it is easy to apply and it provides an exact fit to the current term structure of interest rates

- Disadvantage: little flexibility in choosing the volatility structure, since all spot and forward rates have the same standard deviation σ

Hull and White Model

- Hull and White proposed a simple generalization of the Vašíček model, with time-dependent reversion level, in which:

$$dr_t = \alpha(\mu_t - r_t)dt + \sigma dW_t$$

σ is the constant instantaneous standard deviation of the short rate

μ is a deterministic function of time

The model can be generalized to have *α* and σ deterministic but time dependent.

2 Convexity Adjustment

- forward and futures contracts
- forward-futures spread
- CA in Hull-White, Ho-Lee and other models
- our data set
- Euribor and Eurodollar futures quotes
- parameters estimation
- CA for our data set

Forward and Futures Contracts

- **Forward contract**: agreement to pay a predetermined delivery price *K* at a predetermined delivery date *T* for the asset whose price at time *t* is S_t

- **Forward price** $For_S(t,T)$: value of *K* that makes the forward contract have no-arbitrage price zero at time *t*

$$For_t^T = \frac{S_t}{B_t^T}, \qquad 0 \le t \le T$$

where B_t^T is a price of a zero-coupon bond paying *1* at time *T*

- **Futures contract**: cash flow- daily rolling of the forward contract. Agreement to receive/pay the changes in the future price, i.e. $Fut_t^T - Fut_s^T$ as a cash flow

$$Fut_t^T = E^Q\left[S_T \middle| \mathsf{F}_\mathsf{t}\right] \qquad 0 \le t \le T$$

Forward-Futures Spread

- The difference between forward and futures price: given by the "local" covariance between the rate of return on the futures contract and the rate of return on a risk-free pure discount bond

$$
\begin{aligned}
For_0^T - Fut_0^T &= \frac{S_0}{B_0^t} - E^Q S_T \\
&= \frac{1}{B_0^T}\left[S_0 - B_0^T \cdot E^Q S_T\right] \\
&= \frac{1}{B_0^T}\left[E^Q\left[D_T S_T\right] - E^Q D_T \cdot E^Q S_T\right] \\
&= \frac{1}{B_0^T} \cdot \operatorname{cov}\left(D_T, S_T\right)
\end{aligned}
$$

Forward-Futures Spread

- The difference between futures and forward rates: determined by the market's expectations of future changes in rates
  - different interest rate model will lead to different CA

- If deeper analysis will provide a support for non-zero local covariance, the use of the no-arbitrage model may be questioned

- Empirical research done so far:

1. Metal market: significant spread confirmed for silver, results for copper less clear
   silver and platinum- spread also confirmed with the futures prices above the forward
2. Currency market: statistically significant spread is very rare
   most of papers- spread for contracts lasting only few months is sufficiently small to be ignored
   real factors causing the spread: transaction costs, taxes, …
   for increasing life of futures contract the spread increases as well (e.g. Eurodollar, up to 10 years)

CA in Hull-White Model

Risk-neutral process for the short rate *r* is:

$$dr_t = a\left[\frac{\theta_t}{a} - r_t\right]dt + \sigma dW_t$$

We can derive that the amount by which the futures rate should be reduced in order to obtain the forward rate is:

$$\frac{B(t_1,t_2)}{(t_2-t_1)}\left[B(t_1,t_2)\cdot(1-e^{-2at_1})+2aB(0,t_1)^2\right]\frac{\sigma^2}{4a}$$

with $$B(t_1,t_2)=\frac{1-e^{-a(t_2-t_1)}}{a}$$

CA in Ho-Lee Model

Risk-neutral process for the short rate *r* is:

$$dr_t = \theta_t dt + \sigma dW_t$$

Bond price and forward rate have the form:

$$P(t,T) = A(t,T) \cdot e^{-r(T-t)}$$

$$f(t,t_1,t_2) = \frac{1}{t_2 - t_1} \ln \frac{P(t,t_1)}{P(t,t_2)}$$

with some deterministic function A(t,T). Using Itô's lemma we obtain:

$$dP(t,T) = r_t P(t,T) dt + (T-t)\sigma P(t,T) dW_t$$

$$df(t,t_1,t_2) = \frac{\sigma (t_2 - t)^2 - \sigma^2 (t_1 - t)^2}{2(t_2 - t_1)} dt + \sigma dW_t$$

CA in Ho-Lee Model

- we consider our model in the risk-neutral world □
  the expected value of the spot rate = futures rate □
  CA = expected change in the forward rate between times *0* and t_1

$$CA = \int_0^{t_1} \frac{\sigma^2 (t_2 - t)^2 - \sigma^2 (t_1 - t)^2}{2(t_2 - t_1)} dt$$

$$= \frac{\sigma^2}{2} \int_0^{t_1} (t_2 - 2t + t_1) dt$$

$$= \frac{\sigma^2 t_1 t_2}{2}$$

Our Data Set

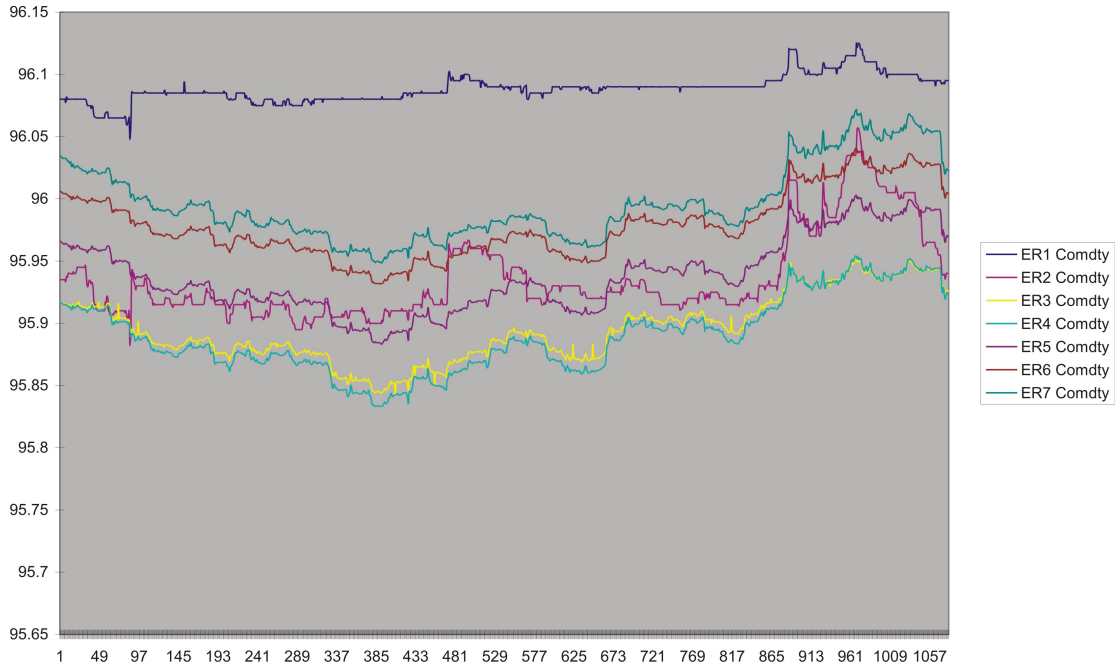
- data set from year 2007 (provided by Bloomberg L.P.), measured between time 08/01/07 and 09/03/07
- intraday 3-month futures prices quotes on EUR - Euribor and USD - Eurodollar currencies
- 7 different maturities: 19/03/07, 18/06/07, 17/09/07, 17/12/07, 17/03/08, 16/06/08 and 15/09/08 – here considered as some of the last maturities before the great crisis appearance
- *MID Price*: the observed bid price plus one-half the bid-ask spread (set as a fair price in our calculations)

$$FutQuote_{e_i}, FutQuote_{u_j}$$

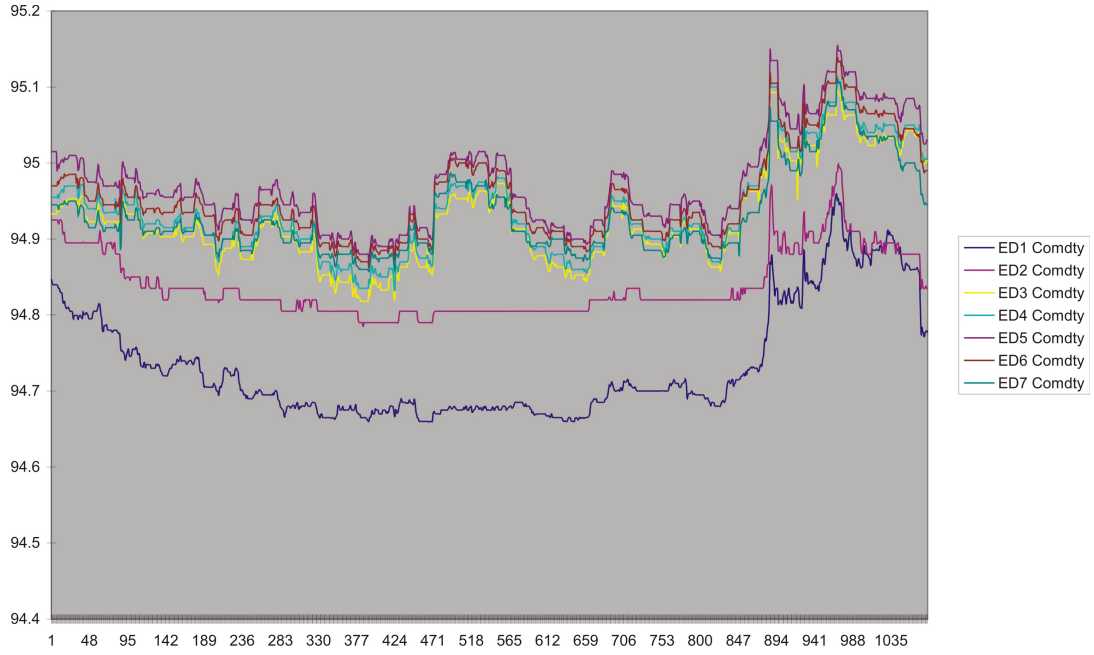
$$i, j = 1,...,7$$

- futures discrete rates and futures rates with continuous compounding are:

$$FutD = 1 - \frac{FutQuote}{100}, \quad FutC = 1 - \frac{\ln(0.25 FutD + 1)}{0.25}.$$



Euribor quotes



Eurodollar quotes

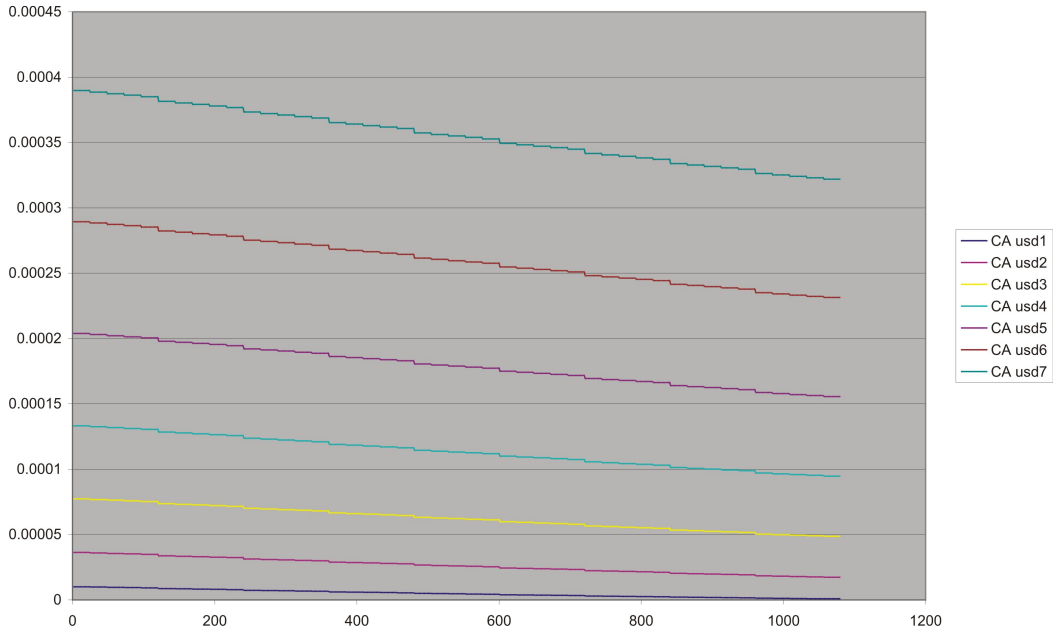
Ho-Lee Model Parameters Estimation

- we denote $FutC_m^d$: futures rate in the time moment determined by the day *d* and its minute *m*
- we need to estimate the standard deviation of daily changes of the futures rates and possibly recompute it to annual basis
- model behind: the minute sequence of the futures rates is assumed to form a random walk. Specially: $FutC_m^d - FutC_m^{d-1}, \quad d = 2,...,D; \quad m = 1,...,M$

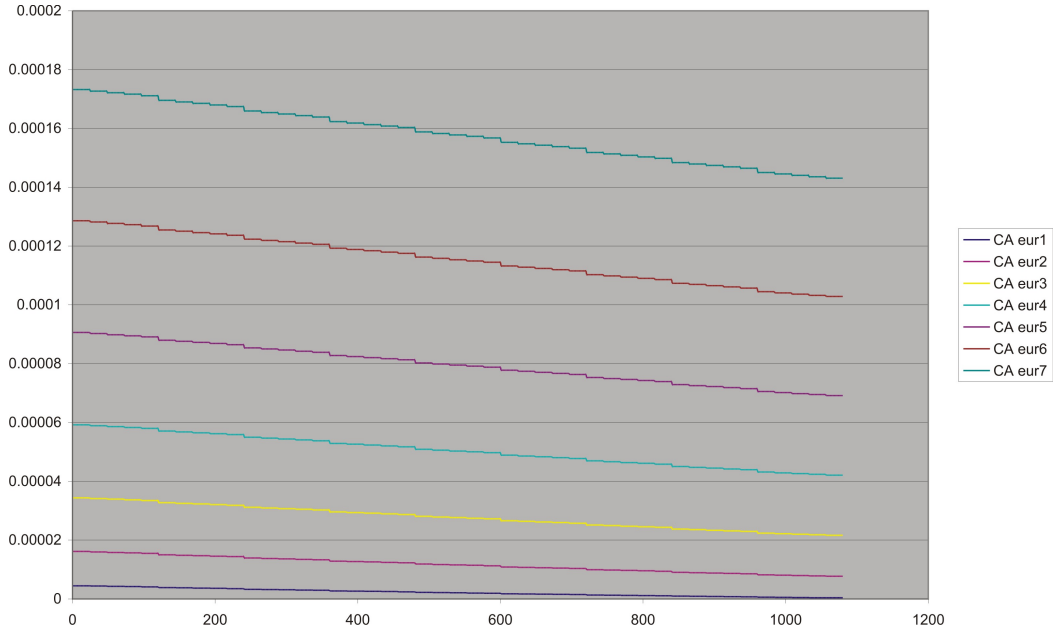
  is a collection of identically distributed variables with zero mean and finite variance σ_{day}^2
- unbiased estimator of σ_{day}^2 : $\hat{\sigma}_{day}^2 = \dfrac{\sum_{d=2}^{D}\sum_{m=1}^{M}\left(FutC_m^d - FutC_m^{d-1}\right)^2}{(D-1)\cdot M}$
- annual estimator: $\hat{\sigma}_{year}^2 = D_{year}\cdot\hat{\sigma}_{day}^2, \qquad D_{year} = 260$

Estimated standard deviation of daily changes of the futures rates

Maturity	EUR	USD
19/03/07	0.97 E-02	1.47 E-02
18/06/07	1.03 E-02	1.31 E-02
17/09/07	1.15 E-02	1.60 E-02
17/12/07	0.85 E-02	1.53 E-02
17/03/08	1.17 E-02	1.61 E-02
16/06/08	1.21 E-02	1.59 E-02
15/09/08	1.13 E-02	1.51 E-02



Convexity Adjustment for the Eurodollar Futures



Convexity Adjustment for the Euribor Futures

CA for Our Data Set: Summary

- as already expected from the model: the exact amount of the adjustment increases with longer maturity
- for a fixed maturity, *CA* in Ho-Lee model is decreasing
- smallest *CA* varies around *0.00001* (*0.001%*) for the Euribor and *0.00004* (*0.004%*) for the Eurodollar
- largest *CA* occurs in case of the last maturity and it is *0.000175* (*0.0175%*) for the Euribor and *0.0004* (*0.004%*) for the Eurodollar
- **implied forward rates**: indicate approximately where short-term rates may be expected to be somewhere in the future

$$For_{e_i} = FutC_{e_i} - CA_{e_i}, \quad i = 1,...,7$$

$$For_{u_j} = FutC_{u_j} - CA_{u_j}, \quad j = 1,...,7$$

3 Arbitrage Analysis

- currency forwards
- construction of the arbitrage
- futures quotes vs. implied quotes
- overview

Currency Forwards

- considering now the most common definition of arbitrage:

  -as a process with positive probability of gain and zero probability of lose-

  we will try construct the arbitrage possibilities using our Euribor and Eurodollar rates and appropriate currency forward rates

- ***currency forward contract***: forward contract that locks in the price at which an entity can buy or sell a currency on a future date (often referred as FX forward or forward outright, here denoted as *ForFX* – Euro FX futures)

- our futures interest rates: 08/01/07 - 09/03/07, with maturities from 19/03/07 until 15/09/08

  for the arbitrage construction we will use the corresponding forward exchanges rates:

- the spot rate, 1 week, 1-, 2-, 3-, 4-, 5-, 6-, 9-, 12-, 15-, 18- month and 2-year rate.

Construction of Arbitrage

- main idea- comparing the two possibilities:

1. having one EUR unit we can first exchange it to the USD using forward exchanges rates and then deposit it with the forward dollar interest rate

2. we can start with deposition using the forward euro interest rate and then exchange it to USD currency using the matching forward exchanges rates.

 at the end of both of these we should arrive to the same amount of USD:

$$ForFX(t)\cdot e^{For_u(t)(T-t)} = e^{For_e(t)(T-t)}\cdot ForFX(T)$$

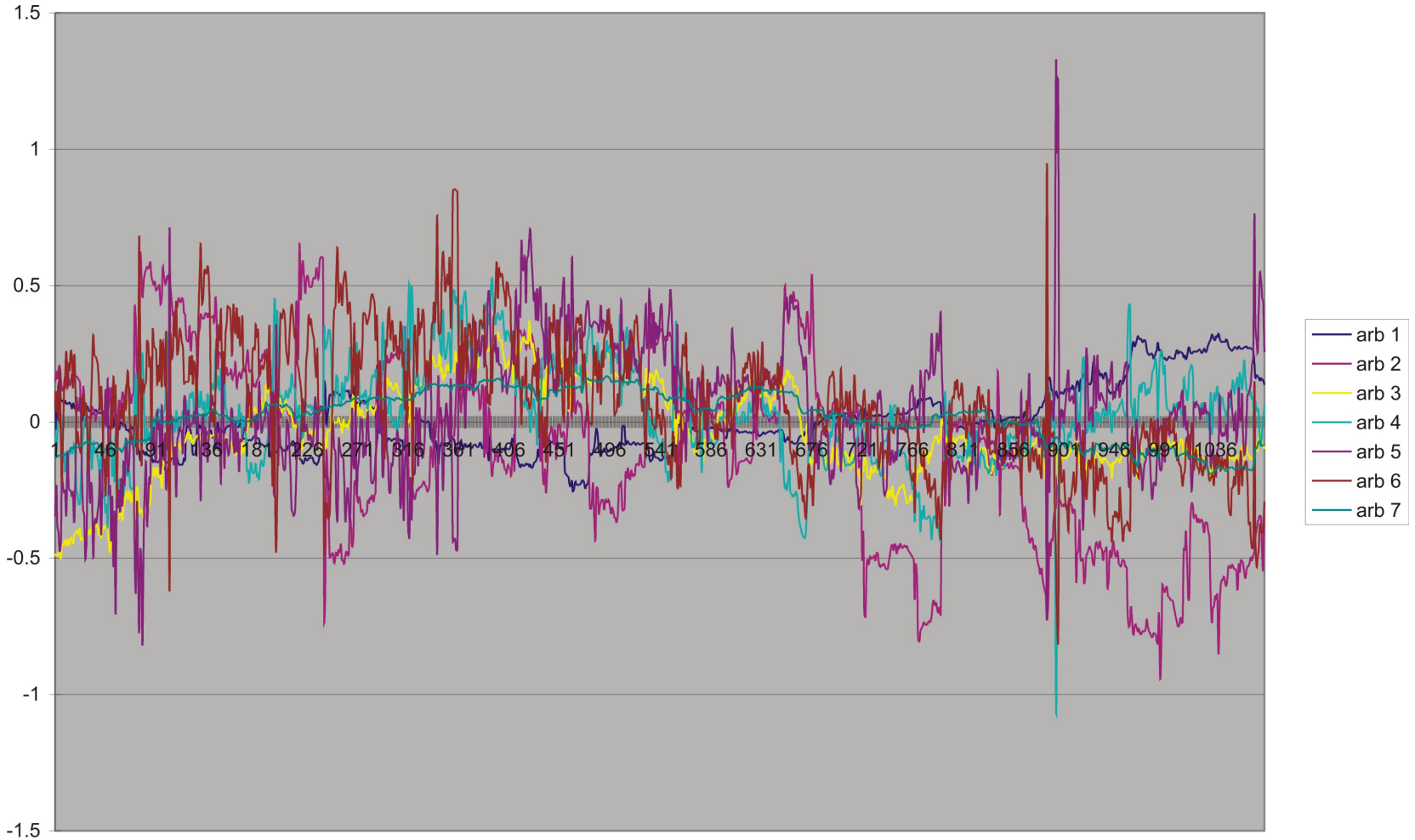
$$For_e(t) = For_u(t) + \frac{1}{0.25}\ln\left(\frac{ForFX(t)}{ForFX(T)}\right)$$

Futures Quotes vs. Implied Quotes

- after computing the implied forward rate for EUR, we can next obtain the implied EUR futures interest rate (by addition of a CA)

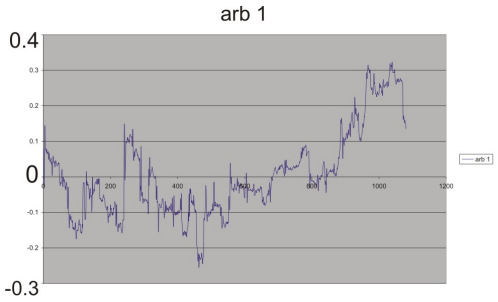
- finally, we will compare the implied Euribor quotes with the real ones, observed from the market

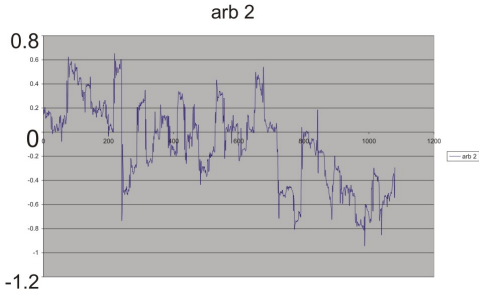
The difference between the real market Euribor quotes and implied Euribor quotes (in bps)

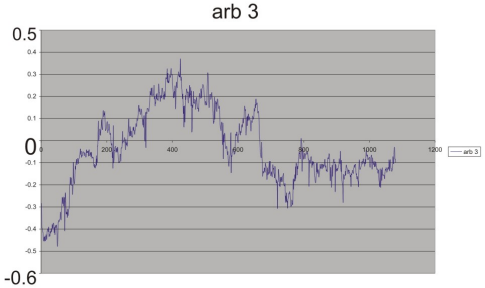


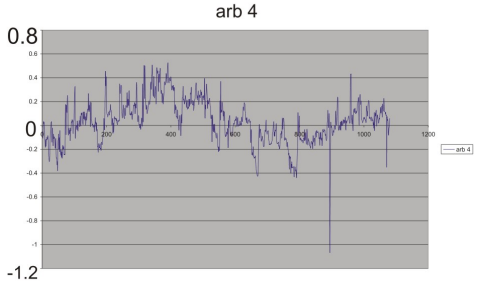
Futures Quotes vs. Implied Quotes

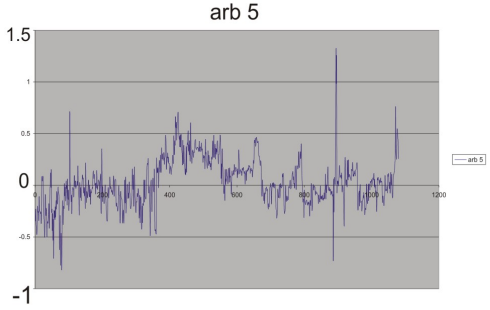
Maturity	Sample Mean	Mean Standard Error	Median	Sample Variance	Max	Min
19/03/07	0.00654	0.00368	-0.01669	0.01466	0.32335	-0.25452
18/06/07	-0.10107	0.01053	0.05179	0.11989	0.64867	-0.94135
17/09/07	-0.03733	0.00528	-0.06815	0.03013	0.36850	-0.49857
17/12/07	0.03386	0.00545	0.03529	0.03216	0.52581	-1.04880
17/03/08	0.03067	0.00690	0.00897	0.05145	1.30983	-0.82011
16/06/08	0.06887	0.00719	0.05593	0.05589	0.94792	-0.80177
15/09/08	0.01909	0.00279	0.02755	0.00839	0.16775	-0.19173

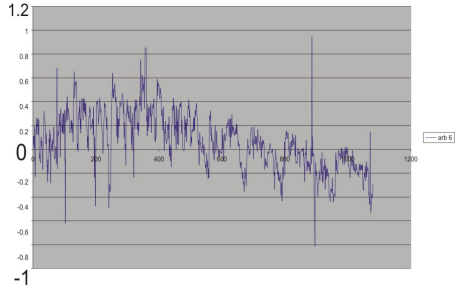


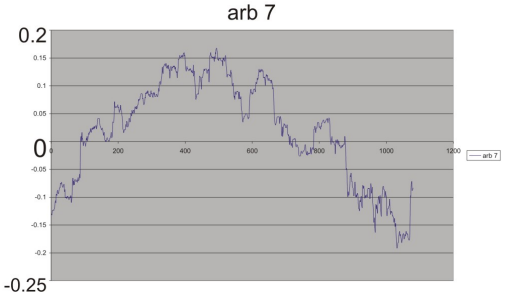












The difference between the real market Euribor quotes and implied Euribor quotes (in bps)

Futures Quotes vs. Implied Quotes Summary

- most striking fact: small sample mean of the arbitrage
- for 2 maturities the sample mean is negative (which indicate that the market quotes are below the implied, calculated ones)
- the rest of the cases: sample mean is positive, but still very close to zero
- trying to test the data for the zero-hypothesis: no moderate statistical sense (most of considerable hypotheses would be strongly rejected according to the high number of observation)
- some certain moments: the trend for some maturities seems to be strongly biased in negative or positive direction

 caused by: imperfect estimation of the appropriate currency forwards,
not all rates were recorded at a same time instant

Thank you for your attention