Worst-Case Approach to Strategic Optimal Portfolio Selection under Transaction Costs and Trading Limits

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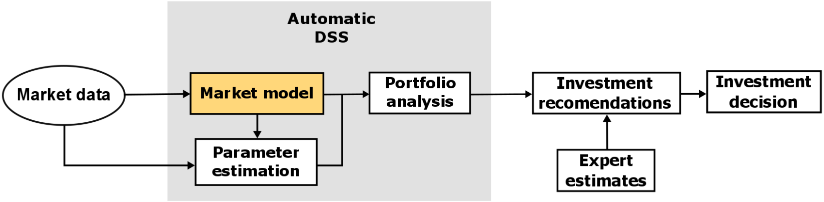
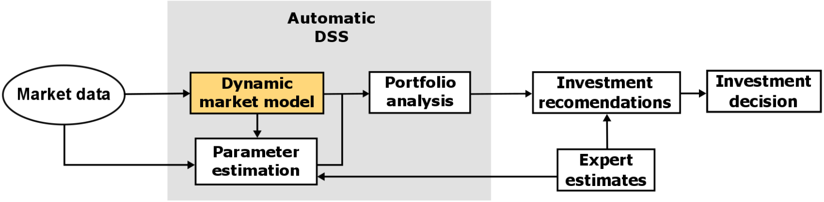
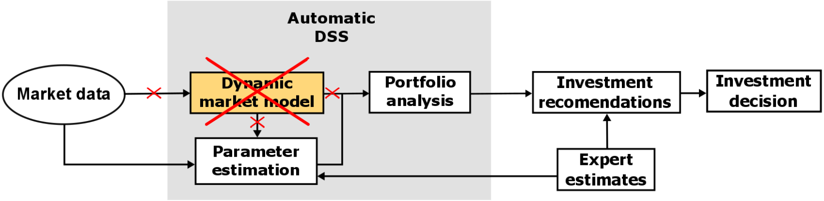


Portfolio analysis, and risk-management in particular, usually relies on the underlying assumptions about the market (ARIMA/GARCH models for returns and VaR, Black-Scholes market model for option pricing etc.)
Portfolio manager considers the results based on the market data analysis and expert analysis.



A more robust process allows to modify assumptions of the market model depending on the market state; an expert can also correct the estimated market parameters if the market data is not representative or insufficient.
Still, part of the analysis depends on some a priori assumptions about the market.



Can make a model-free decision support system which gives investment recommendations?

The answer is YES, but some some model-free assumptions about the market parameters are still required.

Our aim is to formulate a practical portfolio selection problem and provide it with effective numerical solution. Statement of the problem should:

- Be model-free to avoid model-induced errors;
- Allow for uncertain values of the known parameters;
- Consider non-zero transaction costs;
- Include trading limits;
- Consider multi-asset portfolios.

The developed framework should be integrated into the decision support system (DSS) which produces an optimal strategy or a decision map.

Allowed assets and investment policy	— known at the beginning of the strategy.
Trading limits	— known at the beginning of each investment period, limit rules are specified at the beginning of the strategy.
Initial state of the system	— exogenous parameter.
Market model	— specifies the dynamics of the system for a given strategy.
External flows of portfolio assets	— omitted in the current research for simplicity.
Optimal criteria	— a mapping of the system state to the utility value. Specified by investor, can be considered part of the investment policy.

Portfolio is formed at t_0 and rebalanced at fixed moments $t_1, \ldots, t_{N-1}$ (weekly, quarterly etc.). Strategy is evaluated at t_N (finite horizon).

$X_n \in \mathbb{R}^m$, $Y_n \in \mathbb{R}$ — prices of m-dimensional risky asset and risk-free asset at t_n.

$$\Delta Y_n = \xi_n^Y Y_{n-1}, \quad \xi_n^Y = r_n \Delta t_n$$

where r_n is a risk-free rate.
To illustrate the framework, this research considers a general multiplicative dynamics

$$\Delta X_n = \xi_n^X(\Theta_n) X_{n-1}$$

where Θ_n is a vector of market parameters.

Black-Scholes model: $\xi_n^X \sim N(\cdot,\cdot)$

Cox-Ross-Rubinstein model: $\xi_n^X = \begin{cases} u_n & \text{with probability } p_n, \\ d_n & \text{with probability } 1 - p_n \end{cases}$

worst-case approach: any distribution with support K_n and expectation E_n.

For any distribution we can write $\xi_n^X = \underbrace{\mu_n \Delta t_n}_{drift} + \underbrace{\sigma_n \sqrt{\Delta t_n}}_{volatility}$.

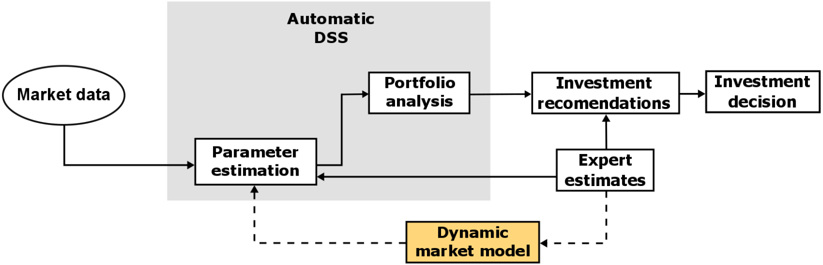
μ_n is the expected value of returns, estimated by the analyst.
σ_n is the deviation from the expected value, estimated by risk-manager.

Below, we assume that r_n and μ_n are estimated by an expert, while σ_n is the only stochastic market parameter.

Worst-case approach: distribution of σ_n is unknown but assumed to have the expected value of E_n and domain $[\underline{\sigma}_n; \overline{\sigma}_n]$.

Parameters of the distribution class can be estimated by an expert, set to reasonable default values (e.g. $E_n = 0$ or $\underline{\sigma}_n = -20\%$, $\overline{\sigma}_n = 20\%$ for MOEX stock market etc.)

Parameters can also be estimated via the **exogenous** market model, which can be rejected or changed if inconsistent with the market data (untangling of DSS and the market model):



H_n^X, H_n^Y — asset volumes.

$W_n^X = {H_n^X}^T X_n$, $W_n^Y = H_n^Y Y_n$ — market values of total risky and risk-free positions.

$W_n = W_n^X + W_n^Y$ — market value of portfolio.

$C_n(H, X)$ — transaction costs function at t_n.

Budget equation:

$$\Delta {H_n^X}^T X_{n-1} + \Delta H_n^Y Y_{n-1} = -C_{n-1}(\Delta H_n^X, X_{n-1})$$

$$\Leftrightarrow$$

$$H_n^Y = Y_{n-1}^{-1}\left(W_{n-1} - {H_n^X}^T X_{n-1} - C_{n-1}(\Delta H_n^X, X_{n-1})\right), \quad n = \overline{1, N}.$$

Our goal is to find an optimal strategy $H^X = \left\{H_n^X\right\}_{n=1}^N$.

Trading limits: $H_n^X \in D_n$, $n = \overline{1, N}$.

Let $\mathcal{S}_n$ be the system state at t_n, $\mathcal{S}_n \mid H_n^X$ be the state given the portfolio H_n.

$\mathcal{S}_0$ is a known initial state.

Classic approach:

$$\mathbb{E}_Q^{\mathcal{S}_0} J(\mathcal{S}_N \mid H_N^X) \to \max_{H^X \in \mathcal{A}}$$

where Q is derived from the specified market model.

Worst-case approach:

$$\inf_{Q \in \mathbb{Q}} \mathbb{E}_Q^{\mathcal{S}_0} J(\mathcal{S}_N \mid H_N^X) \to \max_{H^X \in \mathcal{A}}$$

where $\mathbb{Q}$ is a set of distributions for market parameters $(\Theta_1, \ldots, \Theta_N)^T$ with the expectation E_n and support K_n, $n = \overline{1, N}$.

The optimization problem can be solved by using the dynamic programming principle.
For each state $\mathcal{S}$ of the system at t_{n-1}, we define the value function as

$$V_{n-1}(\mathcal{S}) = \sup_{H^{\mathbf{X}}_{\geq n} \in \mathcal{A}_{\geq n}} \inf_{Q \in \mathbb{Q}} \mathbb{E}^{\mathcal{S}}_{Q} J(\mathcal{S}_N \mid H_N), \quad n = \overline{0, N-1}$$

$$V_N(\mathcal{S}) = J(\mathcal{S}).$$

The Bellman-Isaacs equation:

$$V_{n-1}(\mathcal{S}) = \sup_{H^{\mathbf{X}}_{\mathbf{n}} \in D_{\mathbf{n}}(\mathcal{S})} \inf_{Q_{\mathbf{n}} \in \mathbb{Q}_{\mathbf{n}}} \mathbb{E}^{\mathcal{S}}_{Q_{\mathbf{n}}} V_n(\mathcal{S}_n \mid H^{X}_n)$$

If $H^{X^*}_n \in \operatorname{Arg\,max}_{Z \in D_{\mathbf{n}}(\mathcal{S})} \inf_{Q_{\mathbf{n}} \in \mathbb{Q}_{\mathbf{n}}} \mathbb{E}^{\mathcal{S}}_{Q_{\mathbf{n}}} V_n(\mathcal{S}_n \mid Z)$ for $n = \overline{1, N}$, then $\left\{H^{X^*}_n\right\}_{n=1}^{N}$ is an optimal strategy.

According to the Bellman-Isaacs equation, to find the optimal strategy at t_n one must find infimum over a set of possible distributions Q_n of market parameters at t_n.

Fortunately, when V_n is concave in Θ_n, the extreme measure Q_n^* is concentrated in $p+1$ points of the known support $\mathcal{K}_n$ where p is the number of the market parameters. If K_n is a polyhedron, Q_n^* is concentrated in $p+1$ *extreme* points of K_n.

Therefore, Q_n^* can be found by solving the $p+1$-dimensional optimization problem in general case, or simply by iterating over combinations of $p+1$ extreme points in polyhedral case.

Concavity in the unknown parameters holds for many practical cases. For example, when

- Prices have multiplicative dynamics;
- Transaction costs function depends on the market value of the deal (e. g. linear costs) and convex (always for order-driven markets);
- Terminal utility function depends on the liquidation value of the portfolio;
- Trading limits at each moment are a combination of linear constraints, e. g. of the form

$$Z : Z^T X_n \leq \beta_n W_n \qquad \text{or} \qquad Z : |Z|^T X_n \leq \beta_n W_n,$$

where W_n is the portfolio market capital at t_n, U_n is a matrix.

Let

$$\begin{aligned} s_{n+1} &= I + \mu_{n+1}\Delta t_{n+1} + \sigma_{n+1}\sqrt{\Delta t_{n+1}}, \\ \tilde{r}_{n+1} &= 1 + r_{n+1}\Delta t_{n+1}, \end{aligned}$$

$\mathcal{G}_{n+1}$ be a set of combinations of $m+1$ corners of $\mathcal{K}_{n+1}$, and let $\mathcal{S} = (X, H^X, W^Y)$.

If V_{n+1} is concave in σ_{n+1}, we can write

$$\begin{aligned} V_n(X, H^X, W^Y) = \sup_{Z \in D_{n+1}(X, H^X, W^Y)} \min_{G \in \mathcal{G}_{n+1}} \sum_{i=1}^{m+1} p^i_{n+1}(G) V_{n+1}\Big(s_{n+1}(G_i)X, Z, \\ W^Y \tilde{r}_{n+1} - (Z - H^X)^T X \tilde{r}_{n+1} - C_n(Z - H^X, X)\tilde{r}_{n+1} \Big), \; n < N, \end{aligned}$$

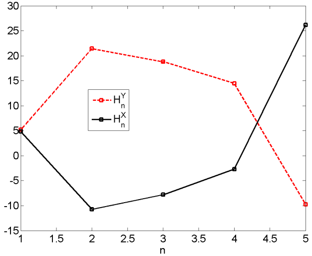
where

$$s_{n+1}(G_i) = I + \mu_{n+1}\Delta t_{n+1} + \mathrm{diag}(G_i)\sqrt{\Delta t_{n+1}},$$

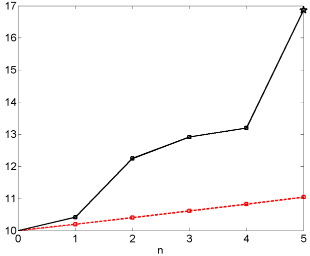
and solve the equation numerically.

Portfolio selection problem

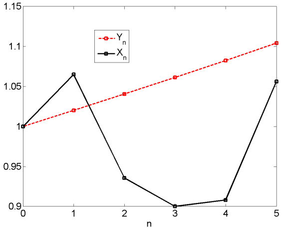
Numeric modeling: $m = 1$, $C_n(H, X) = \lambda |H| X$



Optimal strategy



Portfolio wealth

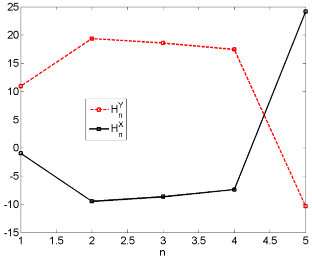


$\lambda = 0$. Due to correct predictions of the analyst, DSS recommendations (considering choice between long and short position) are always correct and portfolio wealth grows during each period.

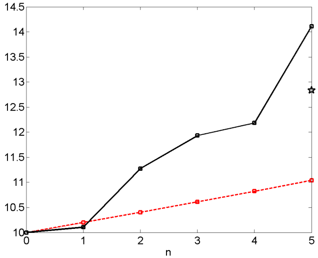
$\longleftarrow$ Prices of risky and risk-free assets.

Portfolio selection problem

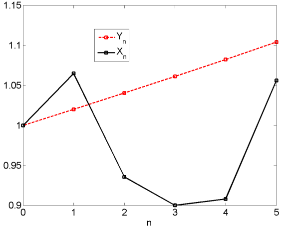
Numeric modeling: $m = 1$, $C_n(H, X) = \lambda |H| X$



Optimal strategy

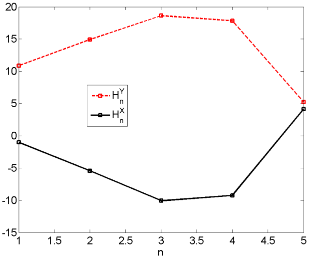


Portfolio wealth

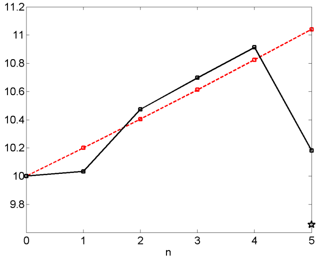


Under non-zero transaction costs $\lambda = 0.05$, trade sizes decrease, lowering total gain of the strategy.

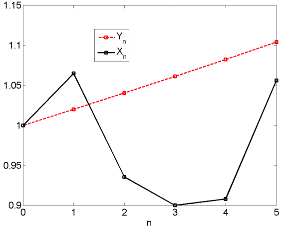
$\longleftarrow$ Prices of risky and risk-free assets.



Optimal strategy



Portfolio wealth



Increasing λ up to 0.12, we see that costs are so high that any risky investments are disadvantageous.

$\longleftarrow$ Prices of risky and risk-free assets.

Conclusion

The worst-case approach is a valid alternative to the classic stochastic portfolio selection approach. Under reasonable conditions the strategy can be found numerically, parallel computing is quite effective.

The main **advantage** is the robustness of the framework: at the beginning of an investment period the required market parameters can be estimated either by an expert, or straight from market data, or by using an appropriate exogenous market model (chosen by an expert). Model risk can thus be minimized.

The main **drawback** is the performance of the strategy since less assumptions about the market are used. Valid assumptions would provide better results. On the other hand, incorrect assumptions could lead to additional losses.

References

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