Mathematical Models of Price Impact and Optimal Portfolio Management in Illiquid Markets

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Can Math Models Really be Trusted?

It's not reasonable to rely on automatic algorithms completely during portfolio management.

But

Due to electronic platforms and HFT it's necessary to react to the information and make decisions quickly.

In many markets liquidity changes over time and unwise trading can lead to great transaction costs.

Can Math Models Really be Trusted?

All portfolio management models are necessary as decision support systems.

More realistic market models lead to more reasonable optimal strategies for current situation at the market.

For the past two decades research in this field has provided complex models that allow for time varying form of limit order book, temporary and permanent price impact, resilience etc.

Main Areas of Development

- Improved microstructure model of the market:
  - Price dynamics (GBM, diffusion or jump-diffusion process);
  - Model of aggregate costs variables such as price impact;
  - Model of the whole limit order book.
- Optimal management problem statement:
  - Optimal criteria;
  - Considering various classes of strategies.

Aspects of Microstructure Modeling: LOB

The most sophisticated and fundamental way of estimating transaction costs is estimating the whole structure of LOB.

Usually market is represented as a complex Poisson process where each event is interpreted as the arrival / liquidation / cancellation of order at specific depth level.

See Cont & Larrard (2012) for similar approach.

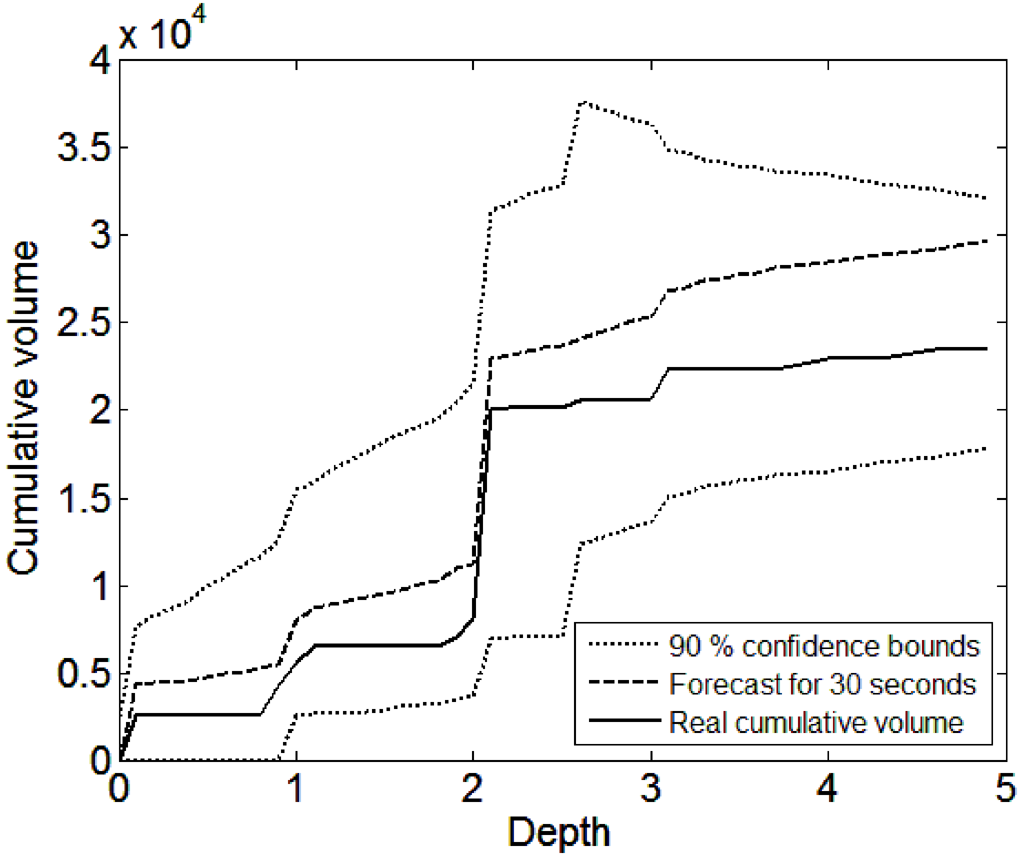
Aspects of Microstructure Modeling: LOB

The main problem is calibration to real data.

The simplest approach , based on empirical observations:

1) orders arrive and leave with different intensities, both depending on depth (|OrderPrice – BestPrice|):

2) each order 's volume is considered a lognormal random variable.



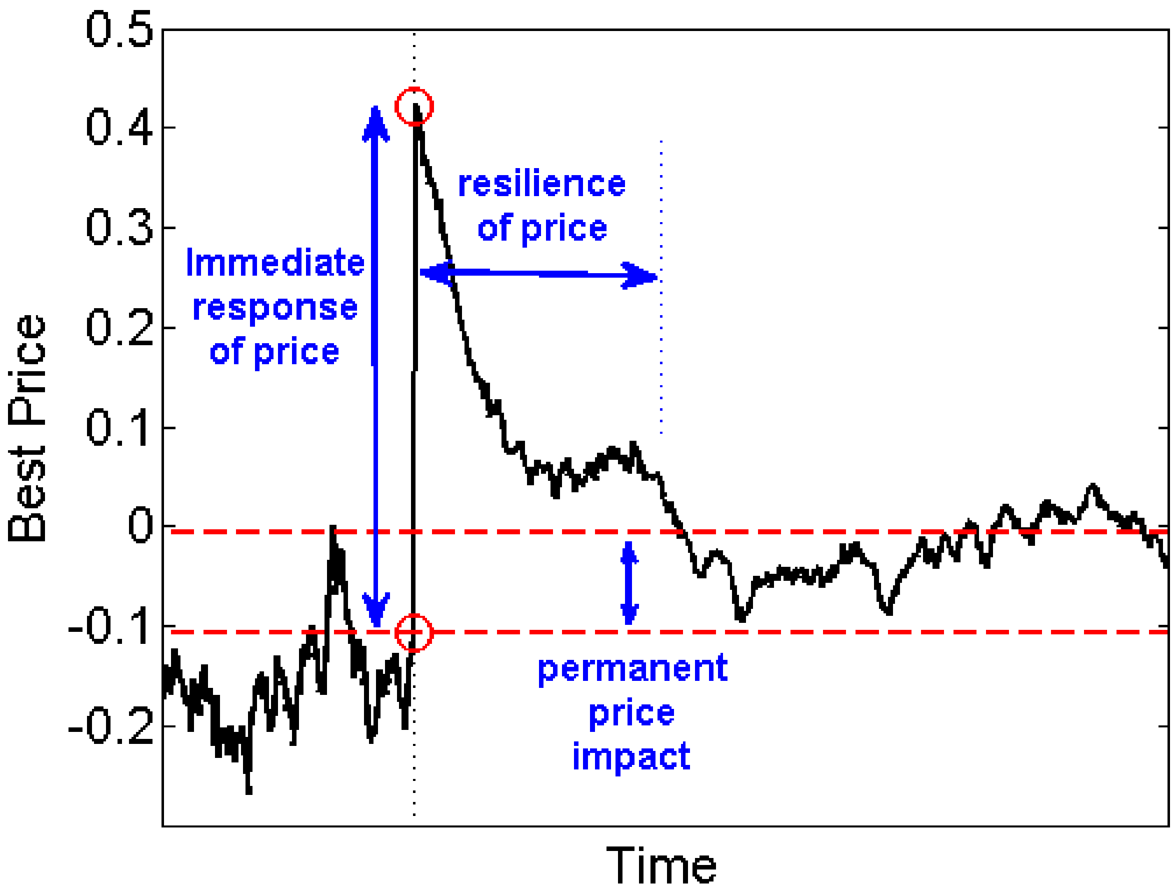
Results are not very good which means that more elaborate assumptions and calibration methods are necessary. Though confidence bounds can be obtained.

Aspects of Microstructure Modeling: LOB

Due to technical difficulties and intention to integrate LOB model into portfolio management optimization problem, a simple a priori form of the book is usually considered.

- Obizhaeva & Wang (2005): flat static structure.
- Alfonsi, Fruth & Schied (2009a): static density (“shadow” LOB), depending on depth.
- Fruth, Schoneborn & Urusov (2011): flat dynamic structure.
- Predoiu, Shaikhet & Shreve (2011): static but most general form of LOB, allows for jumps in cumulative volume, the most realistic model for short periods.

Aspects of Microstructure Modeling: Price Impact



Price Impact = Temporary PI + Permanent PI

Aspects of Microstructure Modeling: Price Impact

PI is not considered in many classical models of portfolio management, such as Merton(1969), Davis & Norman (1990), Oksendal & Sulem (1999), Zakamouline(2002).

Permanent PI:

Linear function of trade volume is usual assumption: Obizhaeva & Wang, Almgren & Chriss, Schied et al.

More complex models: Vath, Mnif & Pham (2005).

Aspects of Microstructure Modeling: Price Impact

Immediate response is usually considered linear in volume.

Smirnov, Naumenko et al. (2010) – polynomial form of response function with stochastic coefficients.

Fruth (2011, PhD thesis) – general law of resilience with stochastic immediate response (several general types of diffusion processes).

Aspects of Microstructure Modeling: Price Impact

Temporary Impact:

- No resilience

  Almgren & Chriss (1999),…

- Exponential resilience $TempImp_t = K_t e^{-\rho(t-t_0)}$

  Obizhaeva & Wang (2005), Gatheral (2011a)

- General law of resilience rate $TempImp_t = K_t e^{-\int_{t_0}^{t} \rho(u)du}$

  Gatheral (2010, 2011b), Alfonsi, Fruth & Schied (2009b), Fruth, Schoneborn & Urusov (2011).

Portfolio Management Problem

Original model by Merton: consider portfolio of risky and risk-free asset, X_t, Y_t - wealth in the assets at time t, c_t - consumption rate.

Market dynamics and trading strategies are continuous, no block trades allowed. Risky asset price follows ordinary GBM. For CRRA utility function $U(x) = x^{\gamma} / \gamma$ maximize expected wealth at terminal period *T* and consumption:

$$J(x,y) = E^{x,y} \int_0^T e^{-\rho t} U(c_t) dt + B(X_T + Y_T, T) \rightarrow \max_{x_t, y_t}$$

Solution: $\pi_t = \dfrac{Y_t}{X_t + Y_t} \equiv const \quad \Rightarrow \quad Y_t = aX_t$ - **Merton line**

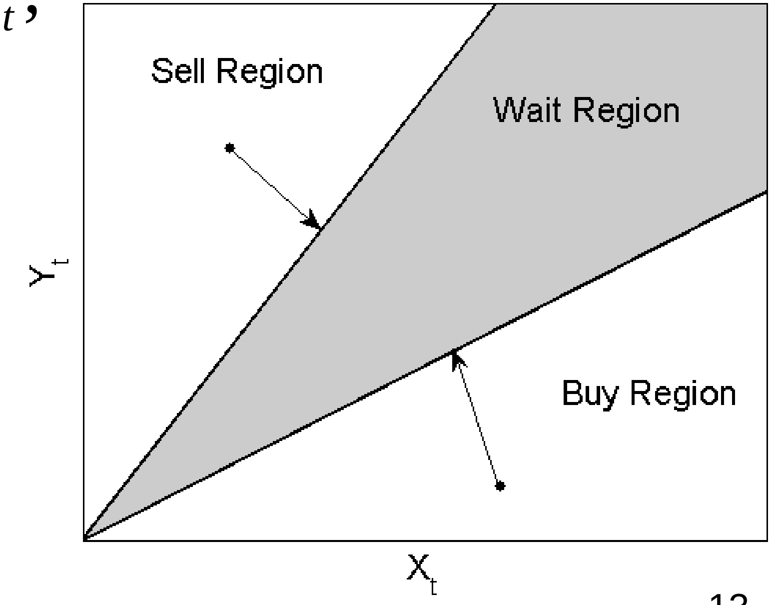
Portfolio Management Problem

Davis-Norman (1990) introduced proportional transaction costs, block trades are allowed. Let L_t, M_t be cumulative purchase and sale of risky asset:

$$\begin{cases} dX_t = (rX_t - c_t)dt - (1+\lambda)dL_t + (1-\mu)dM_t, \\ dY_t = \alpha Y_t dt + \sigma Y_t dw_t + dL_t - dM_t, \\ X_0 = x, Y_0 = y. \end{cases}$$

Shreve & Soner (1994) obtained similar results for infinite horizon.

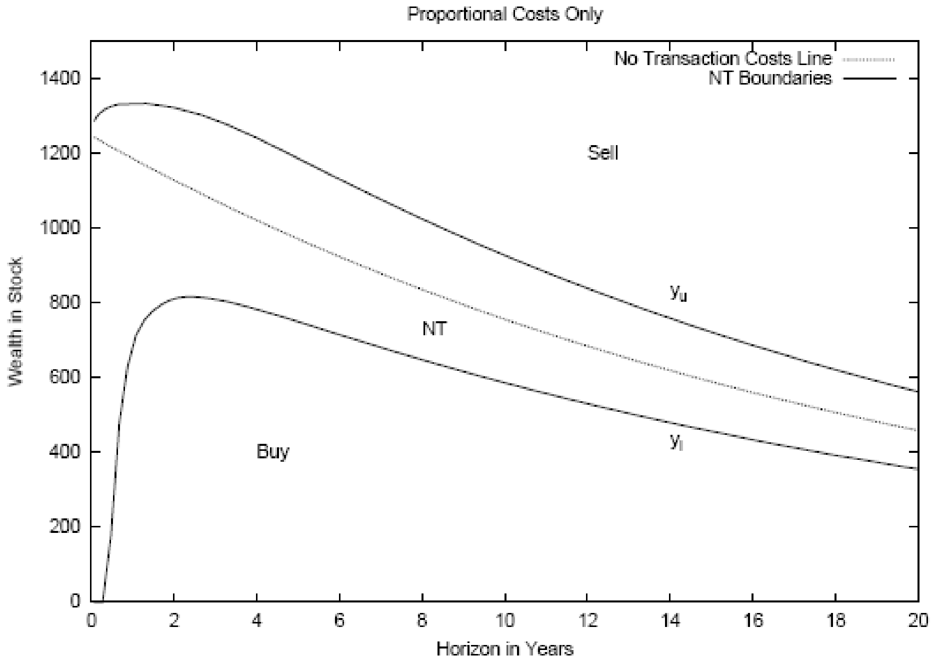
Oksendal & Sulem (1999) obtained cone structure for jump-diffusion price dynamics.



Portfolio Management Problem

Zakamouline (2002) introduced fixed transaction costs (participation fee) and results for CARA utility function.

No analytical solution. Problem is solved numerically with MC methods.



A particular case of the problem, portfolio liquidation, has been researched extensively. In this case we have additional condition for trading strategies: total volume of the position at terminal moment is fixed.

Portfolio Management Problem: Optimal Purchase/Liquidation

Main approaches to define optimality of the liquidation strategy:

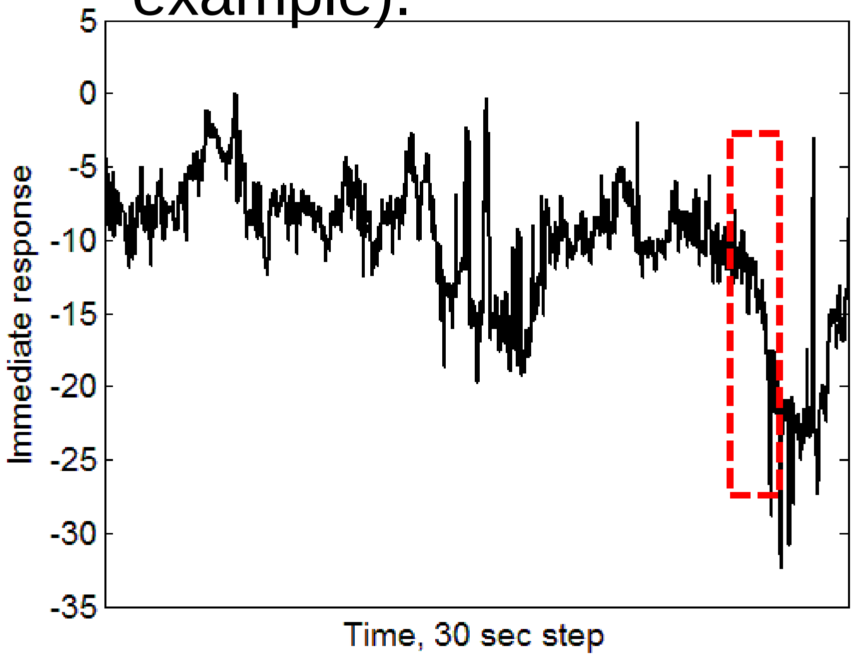
- Tradeoff between market risk and profit;
- Maximization of utility or expected gain;
- Minimization of expected losses due to transaction costs.

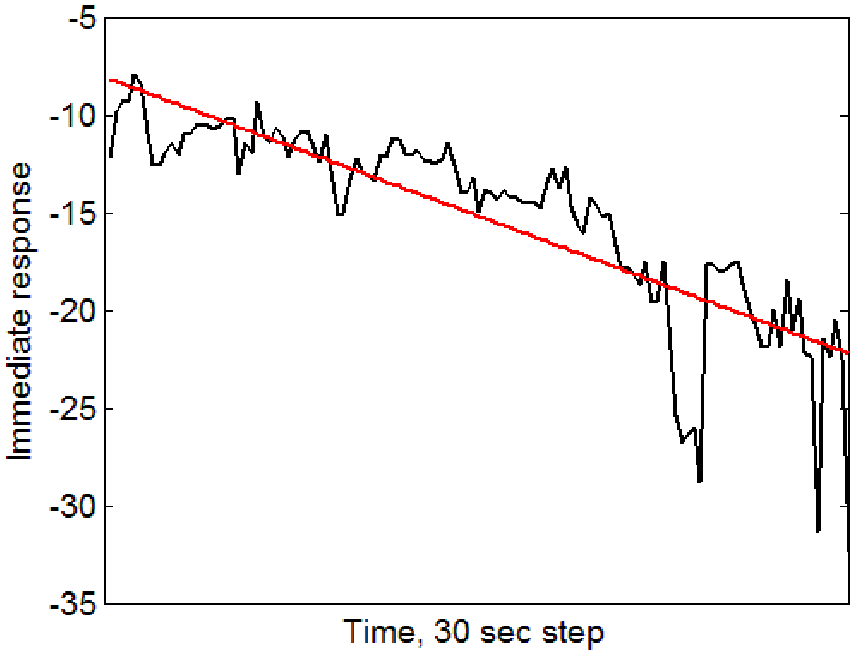
Optimization of expected value is the most popular approach because it often allows to obtain analytical results, such as analytical solution or its properties to simplify numeric procedures.

For risk-management it's also important to consider volatility of results.

Portfolio Management Problem: Comparing Results

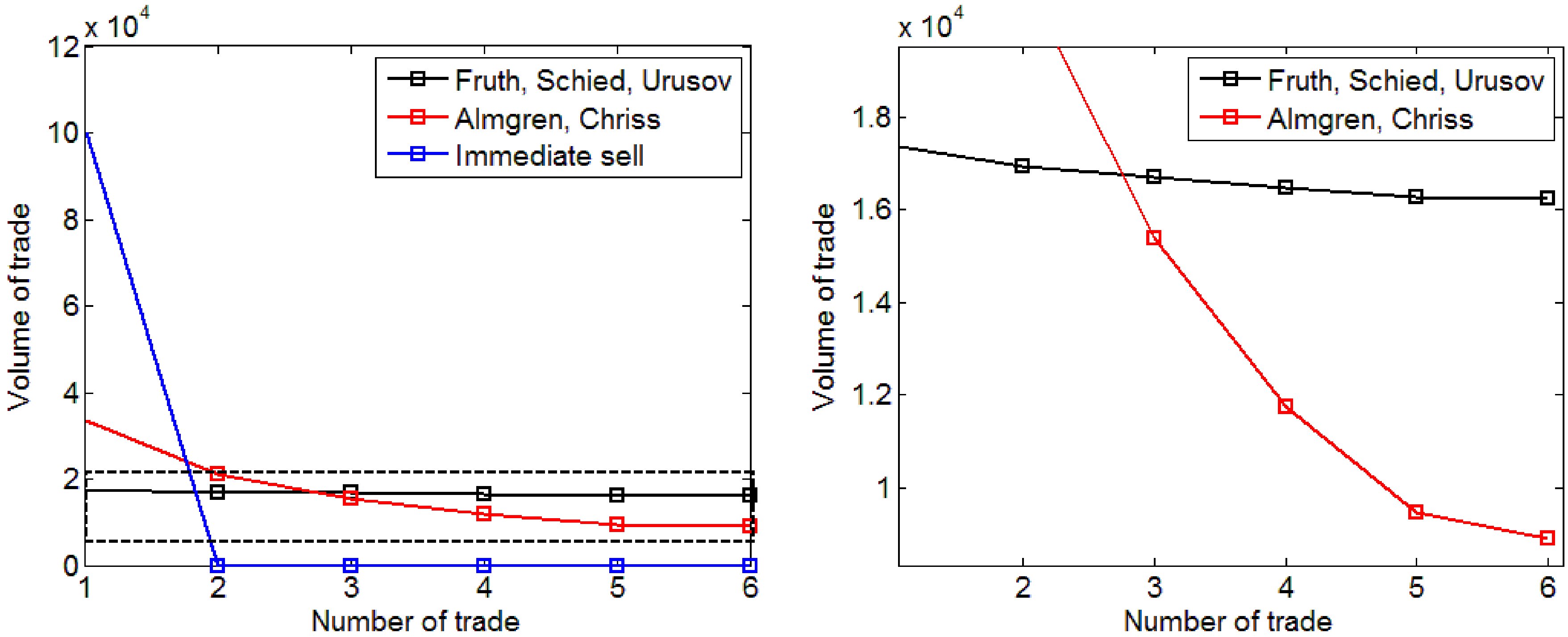
Let’s compare risk-adjusted Naumenko et al. (Almgren & Chriss framework), Fruth, Schied & Urusov strategy with general price impact model and immediate block trade. (Selling portfolio of 100,000 LKOH, 07 Feb 2006, in the example).





Portfolio Management Problem: Comparing Results

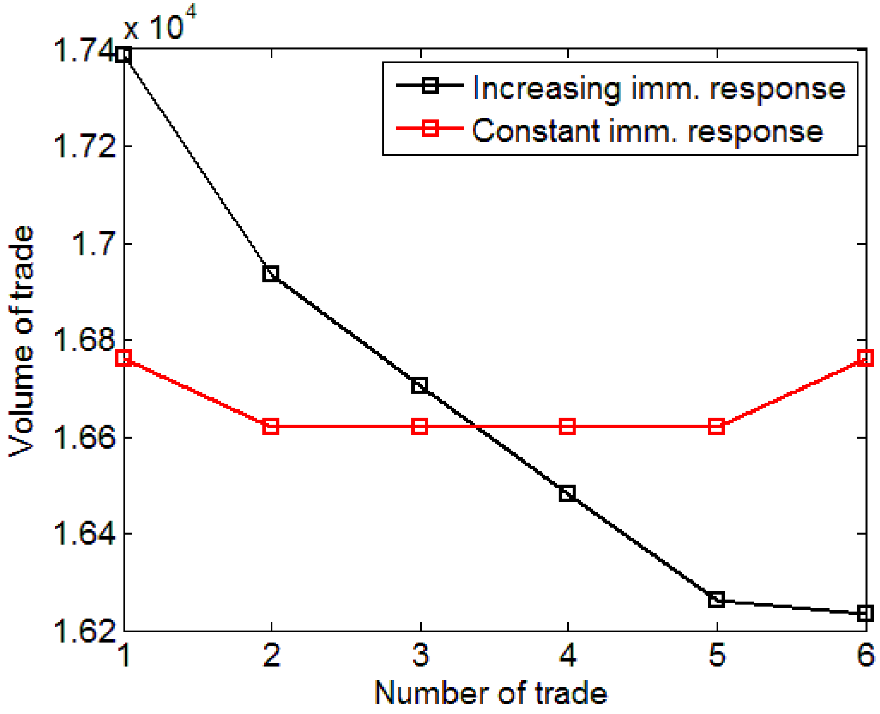
Increase in absolute value of response leads to decreasing aggressiveness of trades in FSU strategy. AG gives the same form due to market risk minimization criterion.



Portfolio Management Problem: Comparing Results

Naumenko et al. strategy's aggressiveness depends on a priori risk aversion parameter, FSU aggressiveness is defined solely by market, because risk is not considered.

On the other hand FSU is better adjusted to a specific form of response (see figure) and can often show better performance.



Liquidation costs:

Immediate block sell < Naumenko et al. < FSU with incr. response

Portfolio Management Problem: Unsolved Issues and Perspectives

- Discrete trading raises a problem of selecting optimal moments to trade.
- The problem has not yet been solved for a general and dynamic form of LOB.
- Optimal portfolio liquidation in continuous time for optimal criterion as in Almgren & Chriss.
- Optimal portfolio management for optimal criterion as in Almgren & Chriss.

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QUESTIONS?

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