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Joint non-parametric approach to credit and interest rate modeling

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Basics of Reduced-Form Credit Risk Modeling

- Modeling of an **unpredictable** time of default.
- The key element is ***the default intensity (hazard rate)***, which is the conditional default probability density.
- Model is to be calibrated to the **market price data**. Corporate fundamentals are **NOT** explicitly taken into account.
- Applied to **pricing** credit risk sensitive instruments.

Pros & Cons of Default Intensity Models

Pros:

- Relative tractability.
- Require little data.
- **Allow modeling interest rate and credit risk in a joint framework.**

Cons:

- Treat default as an absolutely unpredictable event.
- **Results are highly dependent on the chosen specification of the default intensity.**

Interest Rate and Credit Risk Analogies

Discount Function d(t,s)

1. d(t,s1)>d(t,s2) if s1<s2
2. d(t,s)>0
3. d(t,0)=1
4. d(t,s)→0 with s→∞

- **Instantaneous Forward Rate Function**
  - Nonnegative

Survival Probability Function P(t,s)

1. P(t,s1)>=P(t,s2) if s1<s2
2. P(t,s)>0
3. P(t,0)=1
4. P(t,s)→0 with s→∞ (no one lives forever)

- **Default Intensity Function**
  - Nonnegative

Default Intensity Specifications (Deterministic)

- The simplest example is a ***time homogeneous default model.***
- Default intensity is a positive constant $\boldsymbol{\lambda_t}$
- Survival probability:

$$P(t,s)=e^{-\lambda_t s}$$

where ***t*** – current date, ***s*** – term.

- In the moment ***t*** market participants forecast credit quality to remain the same over the time.

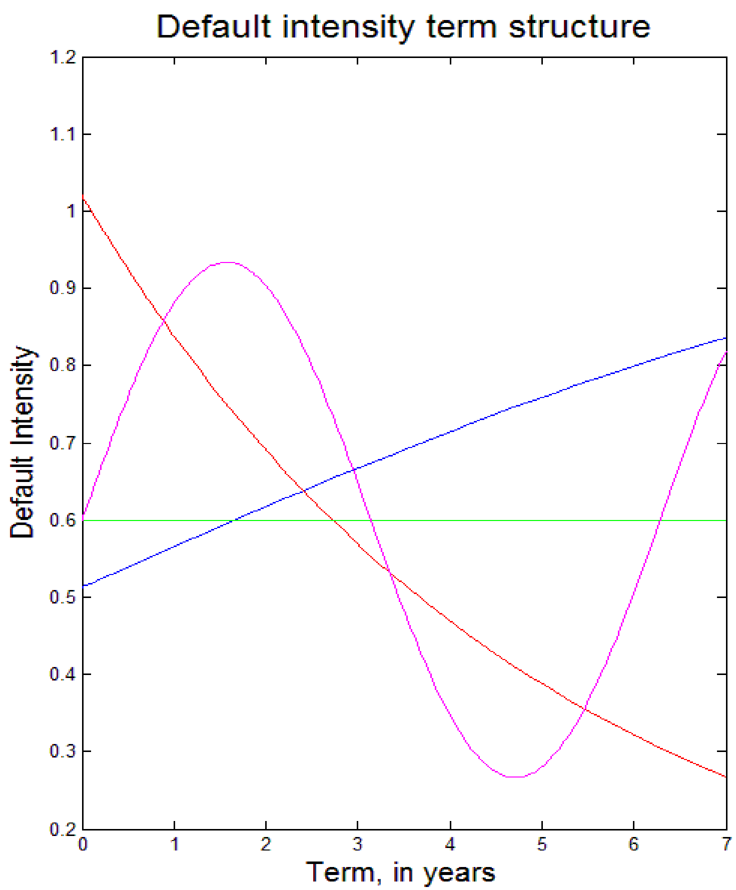
Default Intensity Specifications (Deterministic)

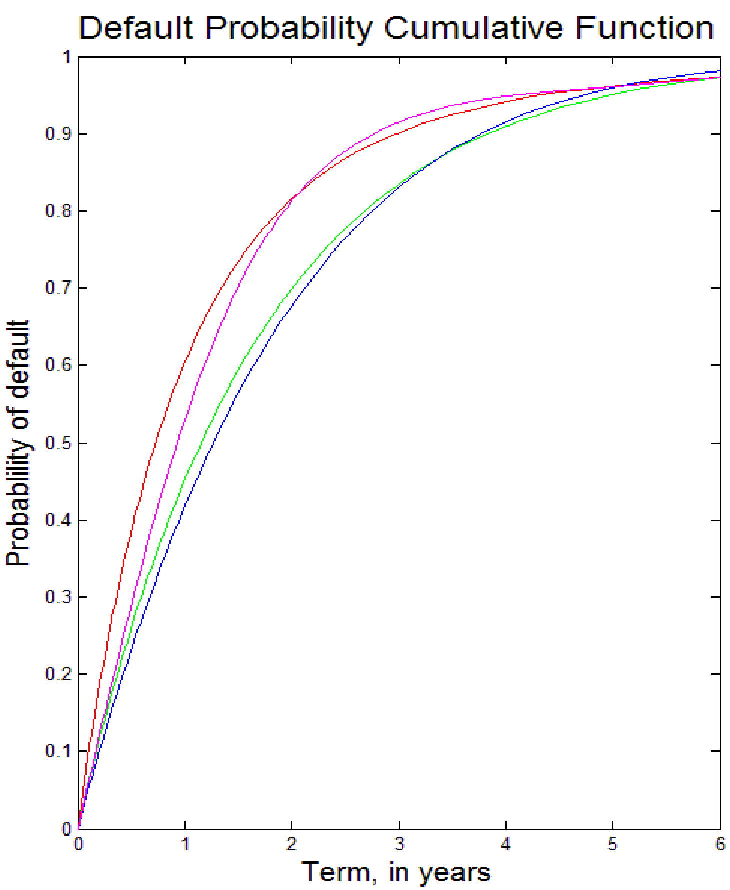
- The next example is a ***time inhomogeneous default model***
- Default intensity term structure is deterministic, but non-constant (*polynomial, exponential, etc.*)
- Survival probability:

$$P(t,s) = e^{-\int_0^s \lambda_t(\tau)d\tau}$$

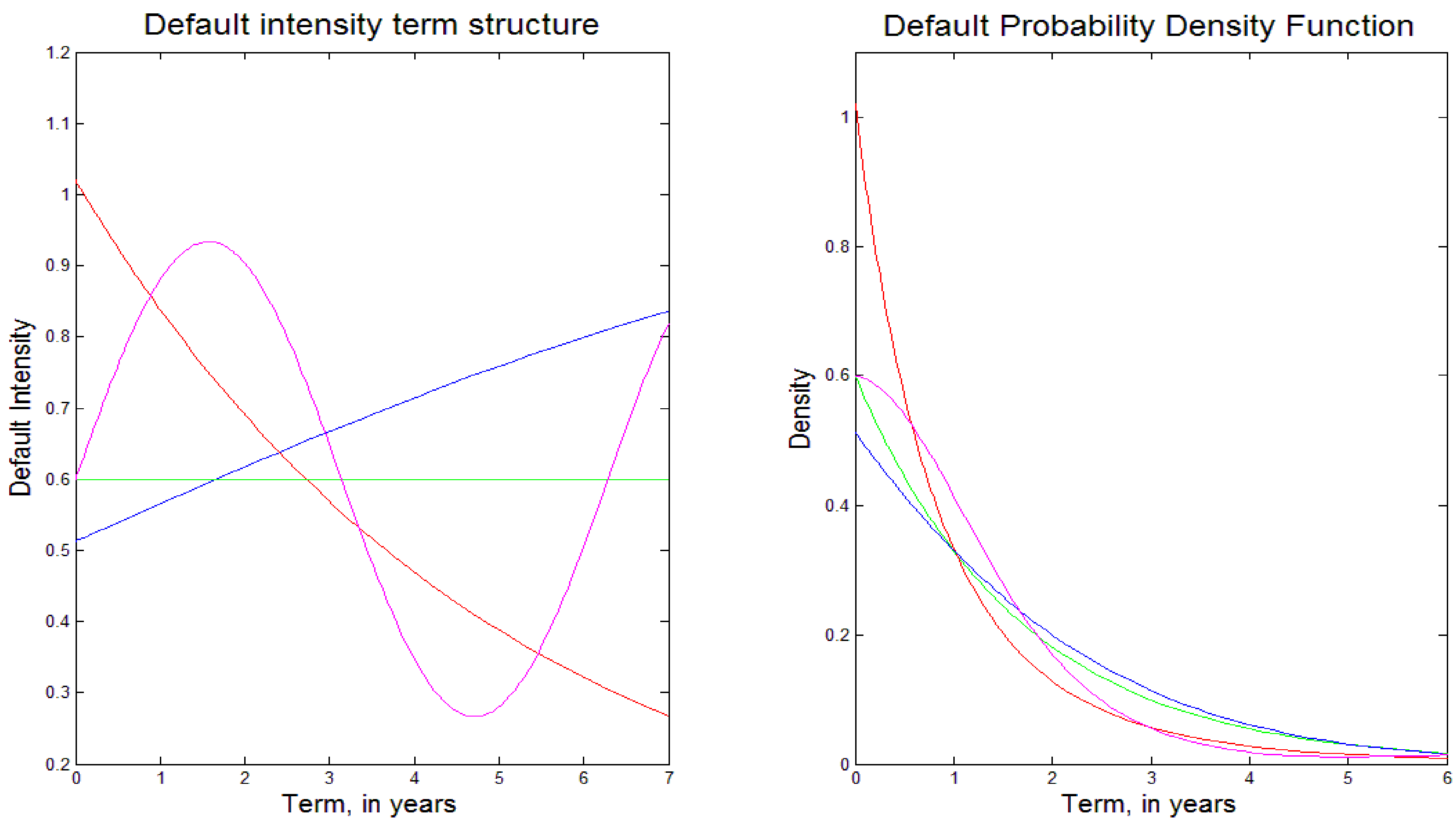
- The form of $\lambda_t(s)$ depends on current perceptions of market participant about future credit quality of the obligor.

Default Intensity vs Probability Function





Default Intensity vs Probability Density



Implementation of Models with Deterministic Specification

- The trivial use of deterministic specification of default intensity is the extraction of risk-neutral hazard rate function via bootstrapping bond or CDS data. Assumed form of hazard rate curve is piecewise constant.
- Bootstrapping methodology is discussed further below.

Default Intensity Specifications (Stochastic)

- Default intensity is assumed to be a stochastic process.
- The market is assumed to be arbitrage-free and complete, therefore a unique risk-neutral probability measure exists, under which default-sensitive assets are priced.
- Risk-neutral survival probability:

$$P(t,s) = E_Q\left(e^{-\int_0^s \lambda_t(\tau)d\tau} \right)$$

- **What process should be chosen?**

Parametric Default Intensity Specifications

- The widest class of parametric specifications of default intensity is **affine model class**.
- Affine models are rather tractable and have quite simple analytical expressions.
- Examples:

✓ Vasicek

✓ Cox-Ingersoll-Ross

✓ Hull-White

Parametric Default Intensity Specifications (continuation)

Vasicek (1977) :

- Hazard rate follows a mean-reverting Brownian motion process.

$$d\lambda_t = \kappa(\mu - \lambda_t)dt + \sigma dW_t$$

- The mean reversion level is constant, thus credit quality does not change in long run.
- Hazard rate volatility is constant as well and independent of hazard rate level.
- The model admits negative hazard rate levels.

Parametric Default Intensity Specifications (continuation)

Cox-Ingersoll-Ross (1985):

$$d\lambda_t = \kappa(\mu - \lambda_t)dt + \sigma\sqrt{\lambda_t}\,dW_t$$

- Hazard rate volatility is proportional to the square root of current hazard rate level.
- Avoids negative hazard rates.
- Still tractable.

Parametric Default Intensity Specifications (continuation)

Hull-White (1990):

- Similar to Vasicek, but the mean reversion level is a function of time.

$$d\lambda_t = \kappa(\mu_t - \alpha\lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t$$

- Model can fit any forward curve, but it is not internally consistent and requires recalibration on a daily basis.

The Heath-Jarrow-Morton framework (1992)

- The entire forward curve depends on a single (or several) stochastic shock, but each instantaneous forward rate has its own sensitivity to this shock.
- An HJM framework may be advanced to an infinite-dimensional extension, which is equivalent to a **non-parametric** specification.

Problems of Joint Estimation of Interest Rate and Credit Risk

1. What specification should be used for interest rates and what for hazard rate?
2. How do interest rates and hazard rates interact?
3. For instruments of a single type (bonds) interest rate and credit premium can **NOT** be separated in a reduced-form model. In order to separate them we have to use several instruments, for example bonds and CDS.
4. Liquidity has a significant impact on bonds prices, therefore ignoring liquidity factor causes errors in interest rates and default probabilities estimates. *see Buhler-Trapp(2006,2008)*

Hazard Rate Term Structure Construction

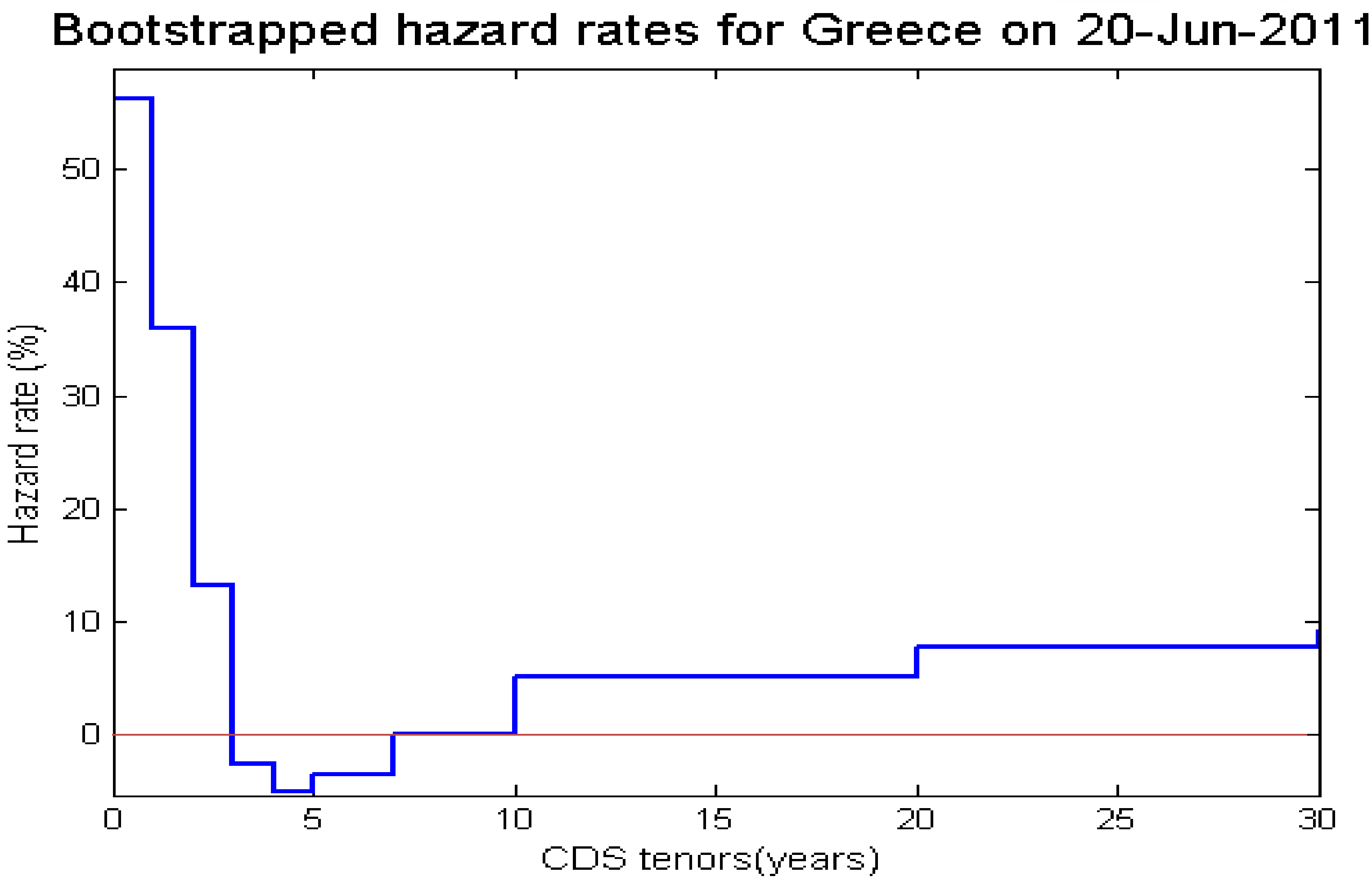
- Use the obtained zero-coupon yield curve to bootstrap default intensities.
- General methodology of bootstrapping hazard rate from CDS data:

1. *Get CDS spreads (or up-fronts) on particular entity for all available tenors and get default-free zero-coupon yield curve;*
2. *Calculate implied hazard rate for the shortest tenor assuming it being constant until CDS maturity;*
3. *Moving to the next longer tenor, find its implied hazard rates for terms between its term to maturity and the term to maturity of previous CDS, assuming hazard rate for shorter terms being obtained on the previous step;*
4. *Recursively calculate entire term structure of hazard rate moving to longer tenors*

Problems with Bootstrapping for Hazard Rate

- CDS premiums are paid on standard dates, thus payment dates for all CDS contracts are perfectly matched, but there are few tenors for particular entity, so in general **CDS data is insufficient** to get satisfactory hazard rate term structure using bootstrapping.
- Assuming piecewise constant form of hazard rate term structure we **artificially increase volatility** of hazard rates in nodes.
- For CDS written on debt of distressed entities (downward sloping CDS spread curve) bootstrapping may yield **negative** hazard rates.
- We illustrate this fact with hazard rate structures bootstrapped from CDS on Greece.

Problems with Bootstrapping



Cox-Ingersoll-Ross Equations for Hazard Rate

- We assume that the risk-free discount function *d(t)* is known.
- We assume the following SDE for the spot default intensity process (risk-neutral):

$$\mathrm{d}\lambda_t = \kappa(\mu - \lambda_t)\,\mathrm{d}t + \nu\sqrt{\lambda_t}\,\mathrm{d}Z_t$$

- The survival probabilities to time t are known to equal

$$P(T) = E\exp\left[-\int_0^t \lambda_\tau d\tau\right] = A(t)\exp\left[-B(t)\lambda_0\right]$$

were A(t) and B(t) have simple analytical expressions.

CDS Pricing Formula

- CDS are priced in terms of par spread:

$$R = \frac{\int_0^T \mathrm{LGD} \cdot d(\tau)\,\mathrm{d}\big(1 - Q(\tau)\big)}{\sum_{i=1}^{N} d(T_i)\alpha(T_{i-1}, T_i)Q(T_i) + \int_0^T d(\tau)\alpha(T_{I(\tau)}, \tau)\,\mathrm{d}\big(1 - Q(\tau)\big)}$$

- where *R* is the CDS par spread, *d(t)* is the discount function, $\alpha(t_1, t_2)$ is the year fraction between t_1 and t_2, and

$$I(\tau) = \max\{T_i : i = 1, \dots, N, T_i < \tau\}$$

How to Get Default-Free Zero-Coupon Yield Curve

- Use one from a “trusted source” such as Bloomberg or Reuters.
- Obtain one from market data using one of the following snapshot methods:

1. *Bootstrapping – too rough and sensitive to errors in data ;*
2. *Parametric (Nelson-Siegel (1987), Svenson (1994)) – produce curves with limited forms spectrum;*
3. *Splines (Smirnov, Zakharov (2003)) – sensitive to errors in data (filtering is needed).*

- Constructed curve is highly dependent on used data (government bonds and interest swaps).

CDS Pricing Assumptions

- LGD is deterministic and constant.
- Liquidity is ignored.
- Often interest rates and default probabilities are assumed to be independent.
- Counterparty risk is NOT taken into account (introduction of CCP).
- In case of credit event CDS is settled at the moment when credit event occurs.

The Fitting Procedure

- We calibrate 4 parameters: *μ, κ, ν,* λ_0 to the observed data via the CDS pricing equation (par spread concept):

$$R = \frac{\int_0^T \mathrm{LGD} \cdot d(\tau)\,\mathrm{d}\left(1 - Q(\tau)\right)}{\sum_{i=1}^{N} d(T_i)\alpha(T_{i-1}, T_i)Q(T_i) + \int_0^T d(\tau)\alpha(T_{I(\tau)}, \tau)\,\mathrm{d}\left(1 - Q(\tau)\right)}$$

where *R* is the CDS par spread, *d(t)* is the discount function, $\alpha(t_1, t_2)$ is the year fraction between t_1 and t_2, and $I(\tau) = \max\{T_i : i = 1, \ldots, N, T_i < \tau\}$

Joint Framework for zero-coupon and hazard rate term structure

- Use a unified model for joint (possibly correlated) dynamics of spot interest rate and spot default intensity, i.e. CIR-like model.
- Snapshot fitting possibilities of simple dynamic models are very limited.
- Systematic errors are introduced from using an inappropriate model for fitting, resulting in misestimating the default probability.

Joint Framework for zero-coupon and hazard rate term structure (cont.)

- Joint stochastic dynamics ***(see Brigo, Mercurio (2006))***:

$$\begin{cases} \mathrm{d}\, r_t = k(\theta - r_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}W_t \\ \mathrm{d}\,\lambda_t = \kappa(\mu - \lambda_t)\,\mathrm{d}t + \nu\sqrt{\lambda_t}\,\mathrm{d}Z_t \end{cases}, \mathbb{E}(\mathrm{d}W_t\,\mathrm{d}Z_t) = \rho\,\mathrm{d}t$$

- If $\rho \neq 0$ then CDS pricing is complicated due to the correlation term:

$$Q(t) : \mathbb{E}\exp\left[-\int_0^t (\lambda_\tau + r_t)\,\mathrm{d}\tau\right] = ?$$

- If $\rho = 0$ then this is just a double CIR model.

Problems with CIR model

Let instantaneous interest rate dynamics be described by the CIR model

$$dr_t = k(\theta - r_t)dt + \sigma dW_t$$

- Snapshot fitting for Eurozone government bonds yields visibly good results.

- Snapshot fitting for CDS prices yields excellent results.

The Proposed Method

- A new infinite-dimensional dynamic model yielding as a by-product a decent snapshot fitting method both for interest rates and default intensities.
- **The implied snapshot fitting method is non-parametric, which allows to eliminate model-inflicted errors.**
- The method is not ad hoc, it is based on a sensible and sufficiently rich stochastic dynamics model.

HJM Equations

- Infinite-dimensional dynamics à-la Filipović (within Heath-Jarrow-Morton framework).
- In Musiela parametrization:

$$dr_t(\cdot) = [Dr_t(\cdot) + \alpha_t(r(\cdot),\cdot)]dt + \sum_{j=1}^{\infty} \tilde{\sigma}_t^j(r_t(\cdot),\cdot)d\beta_t^j$$

- No arbitrage condition:

$$\alpha(r(\cdot),x) = \sum_{j=1}^{0} \tilde{\sigma}^j(r(\cdot),x)\int_o^x \tilde{\sigma}^j(r(\cdot),\tau)d\tau$$

Double HJM Equations

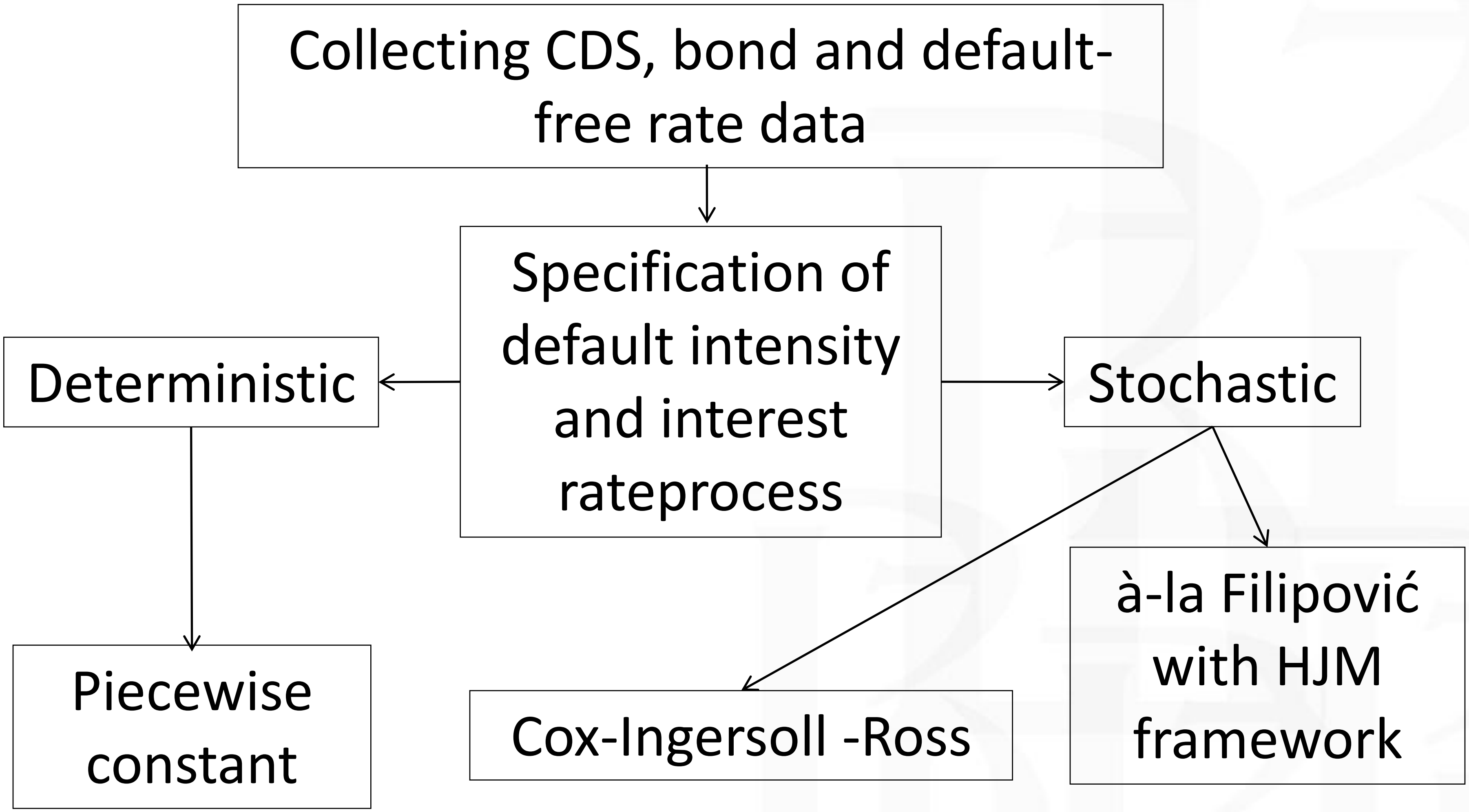
- The joint uncorrelated model

$$\begin{cases} dr_t = (Dr_t + \alpha_t)dt + \sum_{j=1}^{\infty} \tilde{\sigma}_t^j d\beta_t^j \\ d\lambda_t = (D\lambda_t + \tilde{\alpha}_t)dt + \sum_{j=1}^{\infty} \tilde{\nu}_t^j d\tilde{\beta}_t^j \end{cases}$$

gives a nonparametric approach to snapshot yield curve and default intensity fitting.

An infinite-dimensional extension to (Schönbucher, 1998)

The first step: specification of hazard rate



Joint estimation of hazard rates and interest rate

- Find a risk-free spot forward rate curve ***f(t)*** and issuer-specific spot hazard rates $\boldsymbol{h_i(t)}$ such that:
  - CDS quotes are fitted with weights proportional to relative liquidity.
  - Risky bonds prices are fitted with weights proportional to bid-ask spreads.
  - Fitted curves are sufficiently smooth.

Some formulae

$$J_2^k = \sum_T w_T \left[R_2^k - \frac{\int_0^T \mathrm{LGD}\cdot d(\tau)\,\mathrm{d}\big(1-Q_k(\tau)\big)}{\sum_{i=1}^{N} d(T_i)\alpha(T_{i-1},T_i)Q_k(T_i) + \int_0^T d(\tau)\alpha(T_{I(\tau)},\tau)\,\mathrm{d}\big(1-Q_k(\tau)\big)} \right]^2$$

$$J_1^k = \sum_T \frac{1}{ask_T^k - bid_T^k}\left[P_T^K - \left(\sum_{i=1}^{N_{k,T}} d(t_i)F_{i,T}^k Q_k(T_i) + \int_0^T (\text{1-LGD})\cdot d(\tau)\,\mathrm{d}\big(1-Q_k(\tau)\big) \right) \right]^2$$

$$d(t) = \exp\left(-\int_0^T f(\tau)\,\mathrm{d}\tau\right), Q_k(t) = \exp\left(-\int_0^T h_k(\tau)\,\mathrm{d}\tau\right)$$

$$\alpha\int f'(\tau)^2\,\mathrm{d}\tau + \sum_k J_1^k + \beta\sum_k\left(\int h_k'(\tau)^2\,\mathrm{d}\tau\right) + \sum_k J_2^k \to \min$$

The second step: bond model prices calculation and accuracy test

Loss of market value

LGD assumption for bonds

Loss of face value

Bond model prices calculation and risky yield curve construction

Testing accuracy of default intensity process estimate

Comparison with bond market prices

Comparison with bond bid and ask quotes

“Feasibility band” approach

Dataset Description

- **Eurozone sovereign bonds price data**:
- Market price
- Bid & Ask
- Source: Bloomberg

- **Eurozone sovereign CDS price data:**
- Conventional spreads of par spreads
- Source: Reuters

- **Issuers**: Germany, France, Italy, Spain, Ireland, Greece, Portugal
- **Time period**: March 2010 – June 2011

The two methods

CIR

- Simple and tractable.
- Used as a basis best-practice stochastic models.
- Provides a limited, but reasonable spectrum of shapes for yield curves and hazard rate curves.
- Difficult to upgrade to multifactor.

Non-parametric

- Complex.
- Used basically in theory.
- Any reasonable shape for yield and hazard rate curves.
- Easily incorporates multifactor (and even infinite-factor) dynamics.

Our Fitting Method

- As a snapshot projection of our dynamic model one is required to search for a function ***f(t)=g(t)²*** satisfying

$$\alpha\int_0^T (g')^2(\tau)\,\mathrm{d}\tau + \sum_{k=1}^{N}\omega_k\left(\sum_{i=1}^{n}F_{i,k}d_g(t_i)-P_k\right)^2 \to \min_{g(\cdot)},$$

$$\mathrm{d}_g(t)=\exp\left[-\int_0^t g^2(\tau)\,\mathrm{d}\tau\right]$$

where P_k is the price of the k^{th} bond, $F_{i,k}$ is the cash flow on the k^{th} bond at time t_i.

Our Fitting Method

- The solution is an exponential-sinusoidal spline with knots at every cash flow time:

$$f(\tau) = p_{i-1}\phi_{\lambda_i}(t_i - \tau) + p_i\phi_{\lambda_i}(\tau - t_{i-1}), \tau \in [t_{i-1}, t_i),$$

$$\phi_{\lambda_i}(\tau) = \begin{cases} \dfrac{\sinh\sqrt{\lambda_i}\,\tau}{\sinh\sqrt{\lambda_i}\,(t_i - t_{i-1})}, & \lambda_i > 0; \\[2ex] \dfrac{\sin\sqrt{-\lambda_i}\,\tau}{\sin\sqrt{-\lambda_i}\,(t_i - t_{i-1})}, & \lambda_i < 0; \\[2ex] \dfrac{\tau}{t_i - t_{i-1}}, & \lambda_i = 0. \end{cases}$$

Our Fitting Method

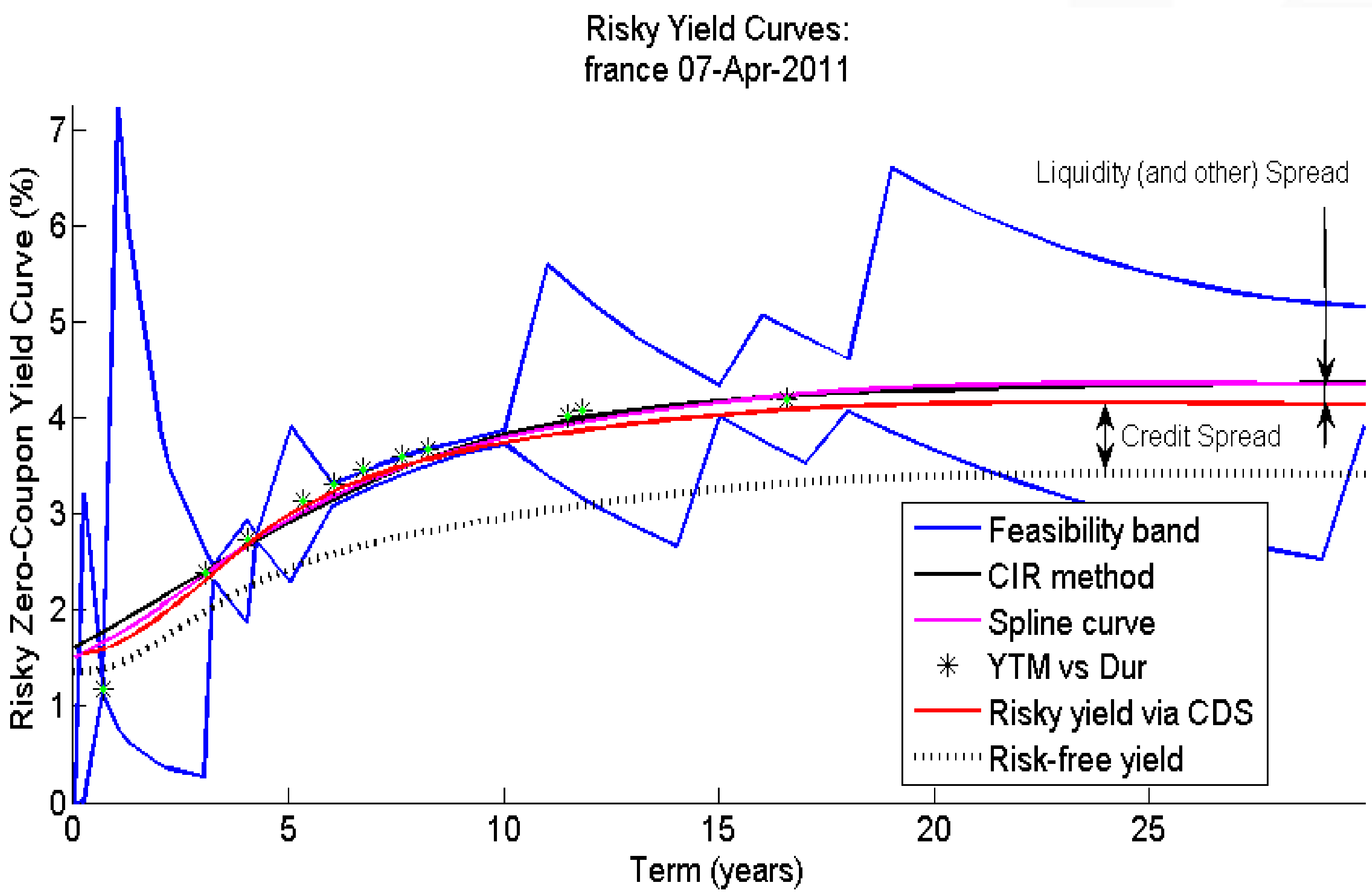
- Ensures positive spot forward rates.
- Ensures continuous and differentiable spot forward rates.
- Takes liquidity (e.g. bid-ask spreads) into account via weighing coefficients w_k.
- Can be fine-tuned to exhibit any desired proportion between smoothness of the forward rate curve and accuracy of replicating bond prices.
- **Does not introduce model-inflicted errors in estimation.**

The Feasibility Band

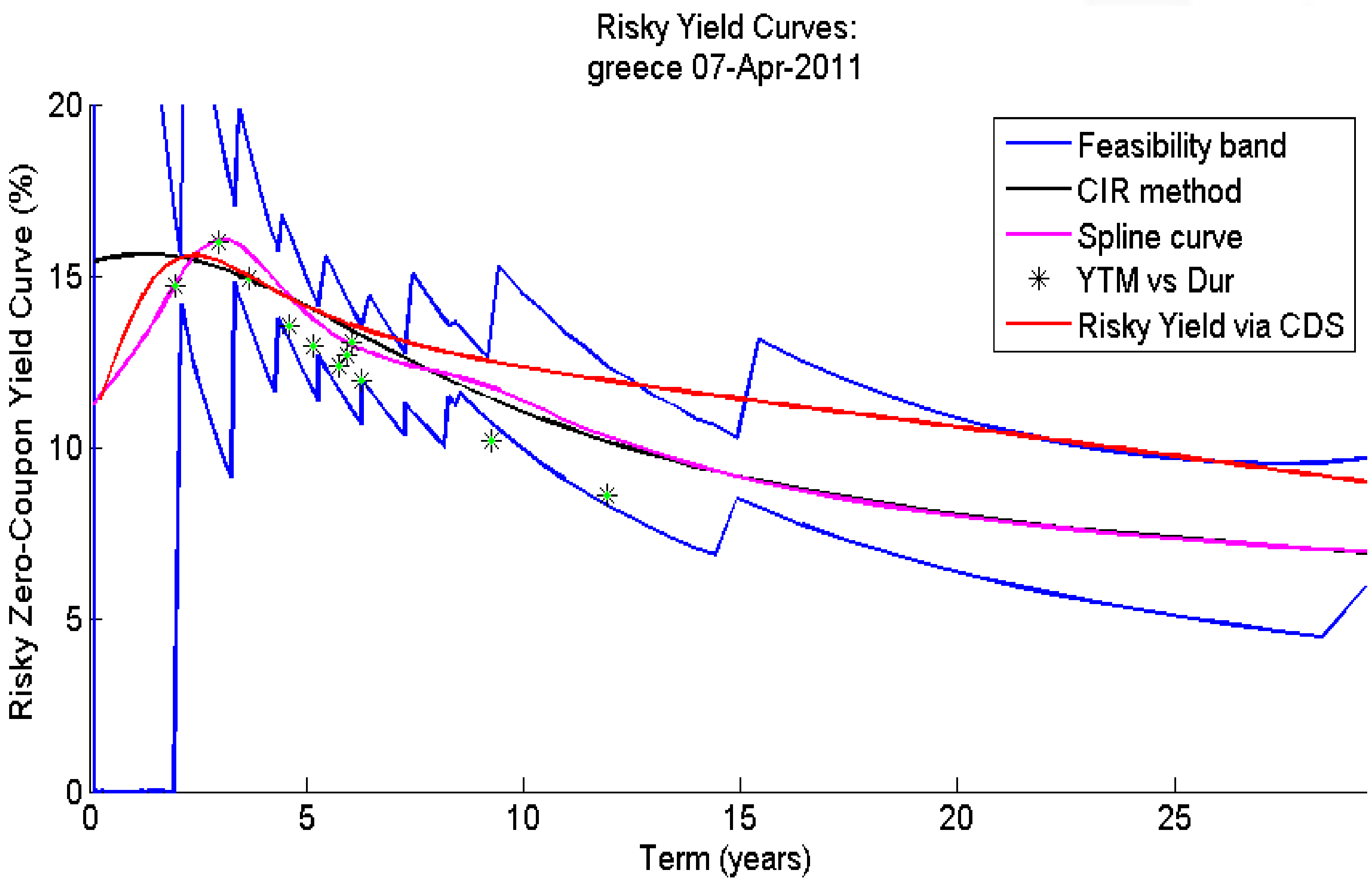
- In order to assess the accuracy of fitting, we employ the notion of a feasibility band.
- It transfers bid-ask bounds to the interest rate domain.

$$
\begin{cases}
r(t_i) = -\frac{1}{t_i} \ln d_i; \\
\mathrm{d}_i \to \max, \min; \\
1 \ge d_1 \ge \ldots \ge d_N \ge 0; \\
ask_k \ge \sum_{i=1}^{N} F_{i,k} d_i \ge bid_k.
\end{cases}
$$

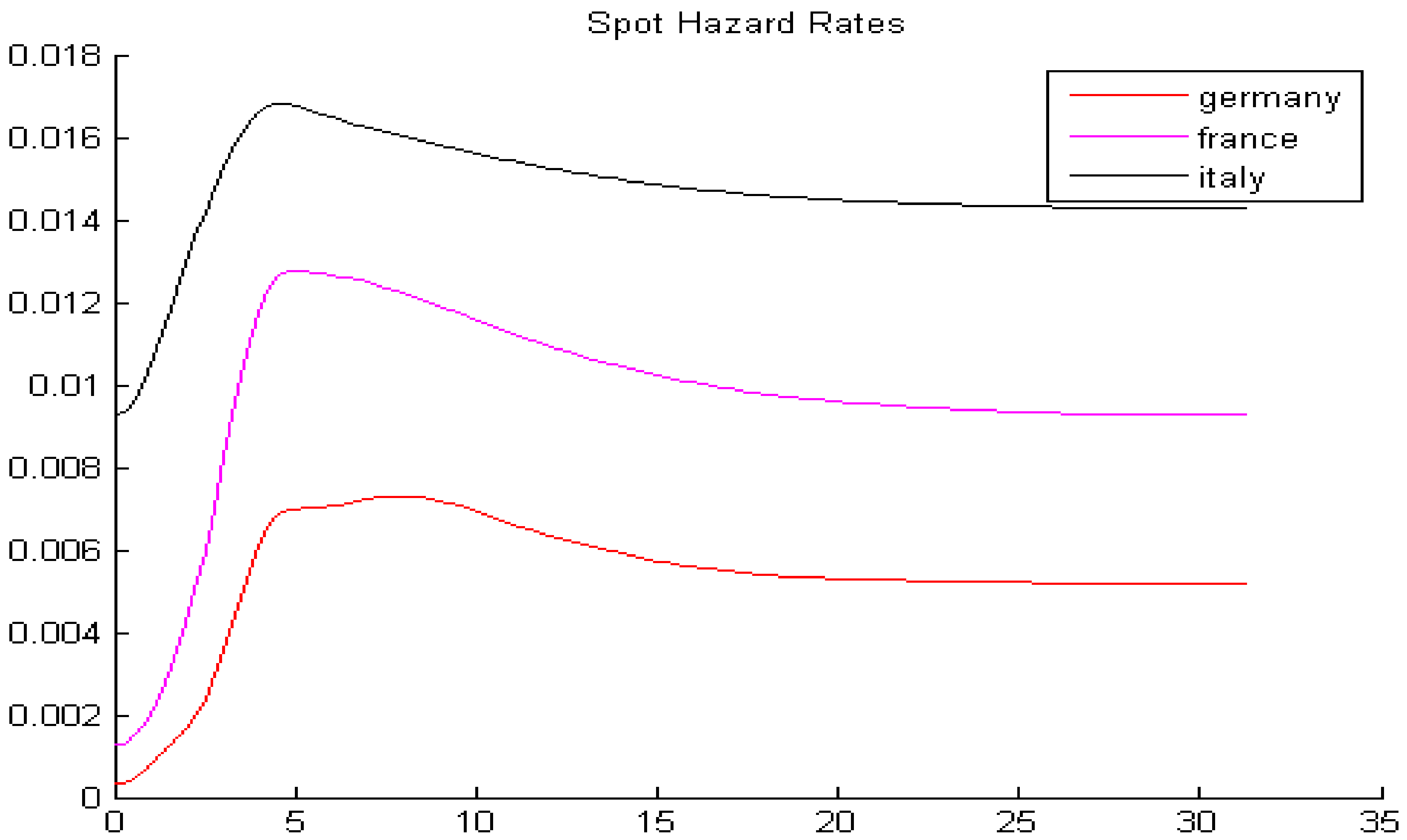
Replication of bond quotes



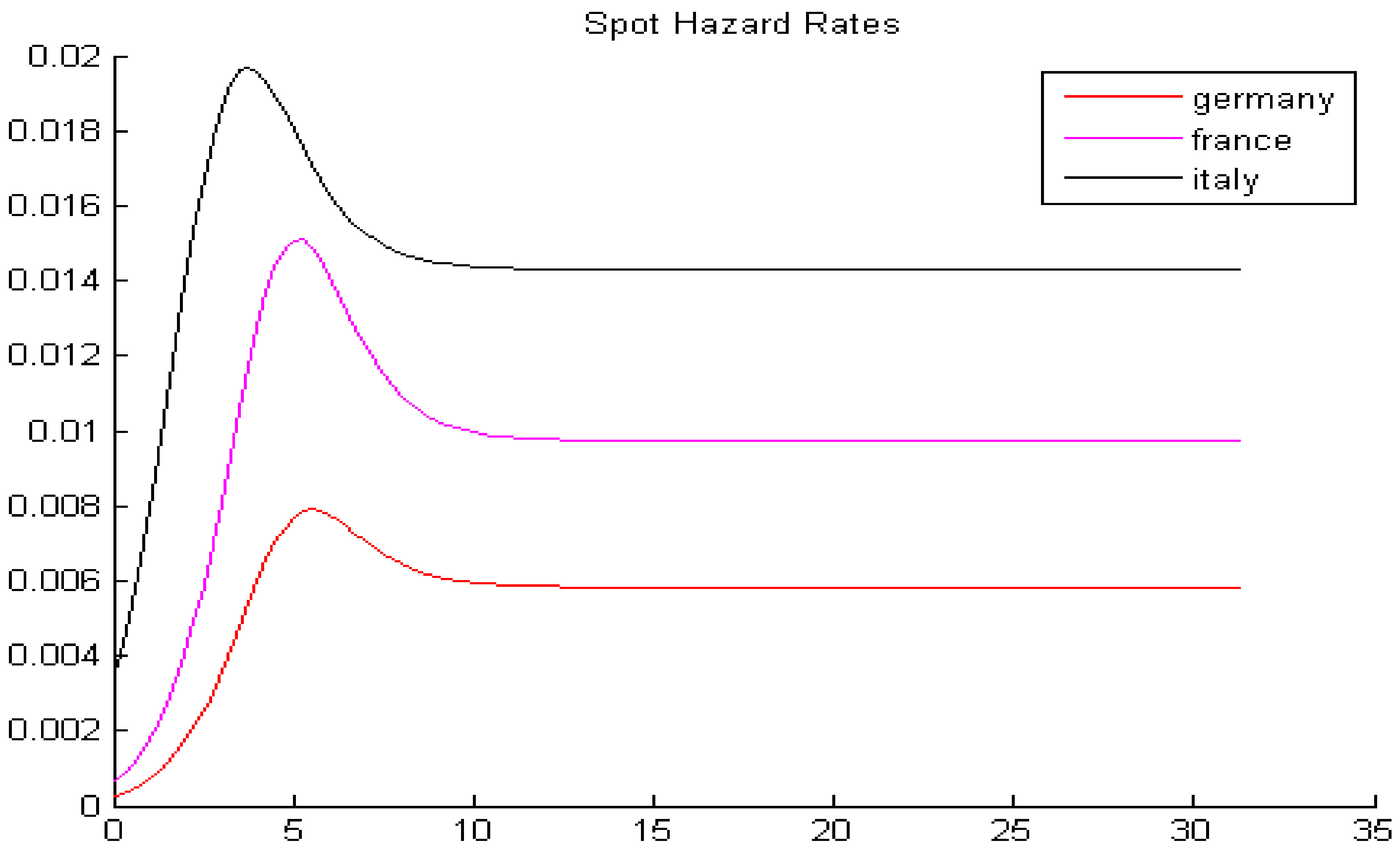
CDS-Implied Yields



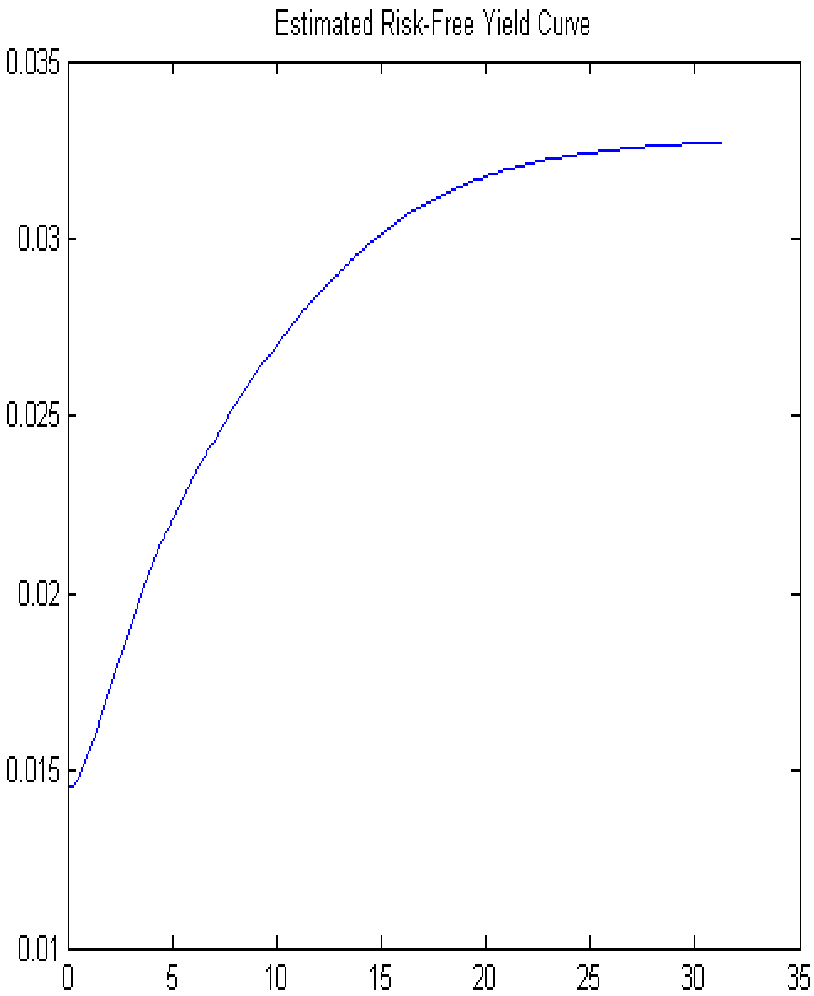
Spot hazard rates

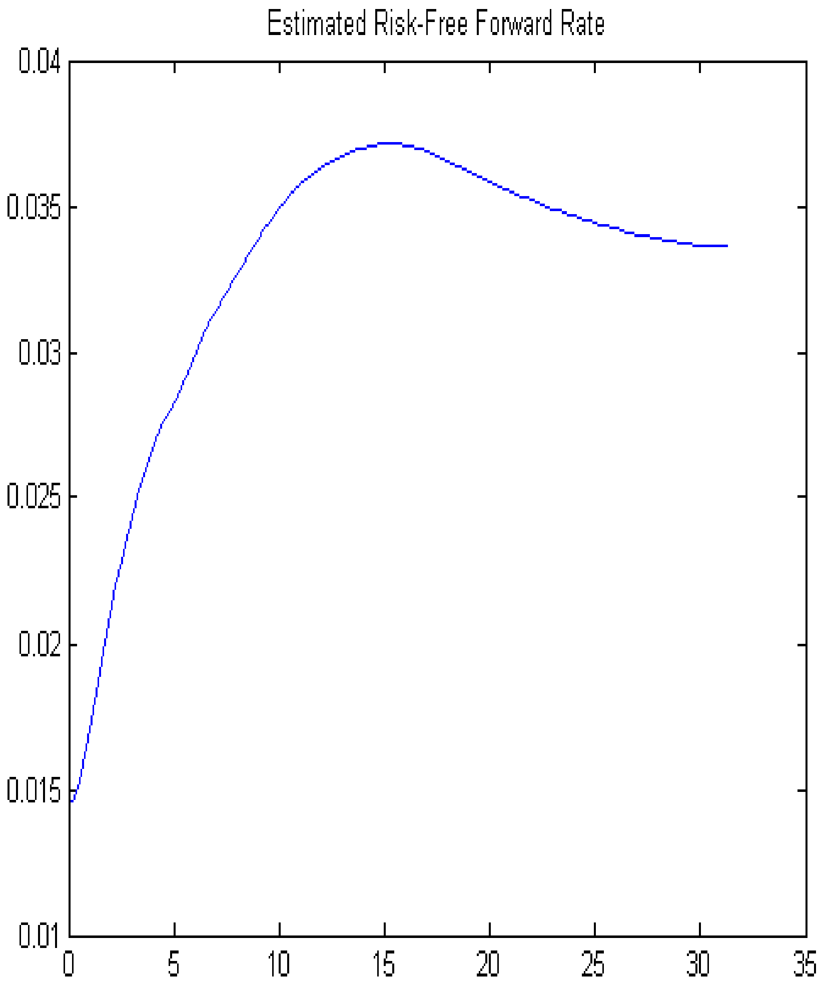


Spot hazard rates: CIR

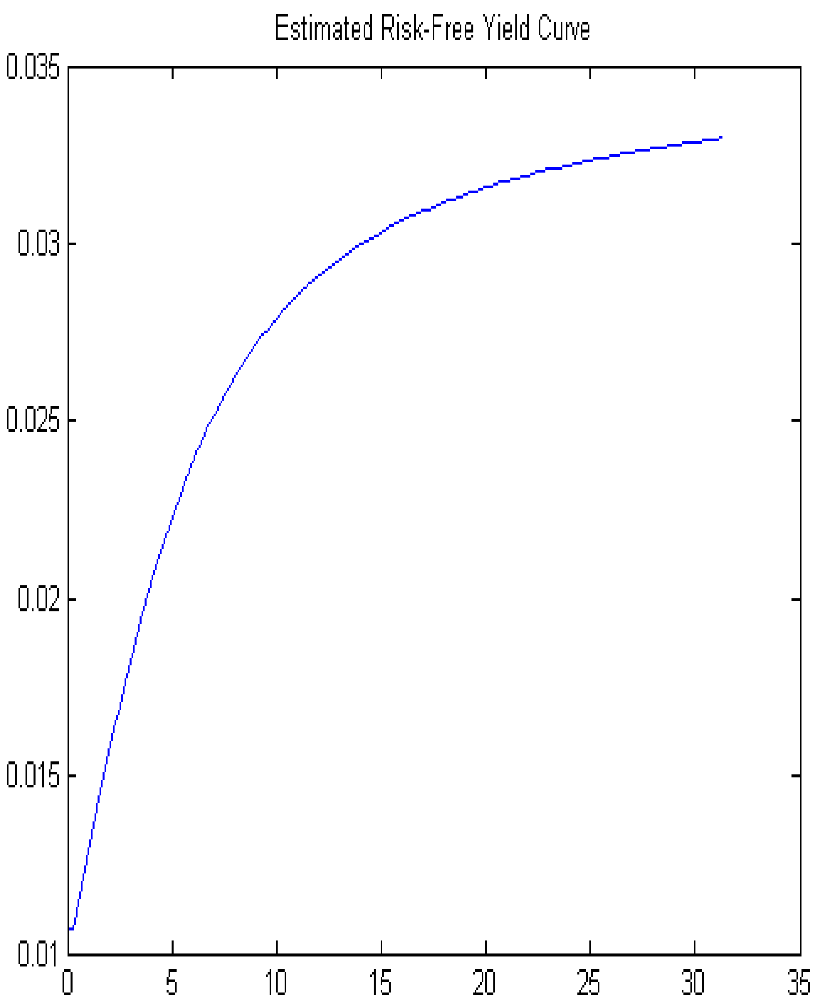


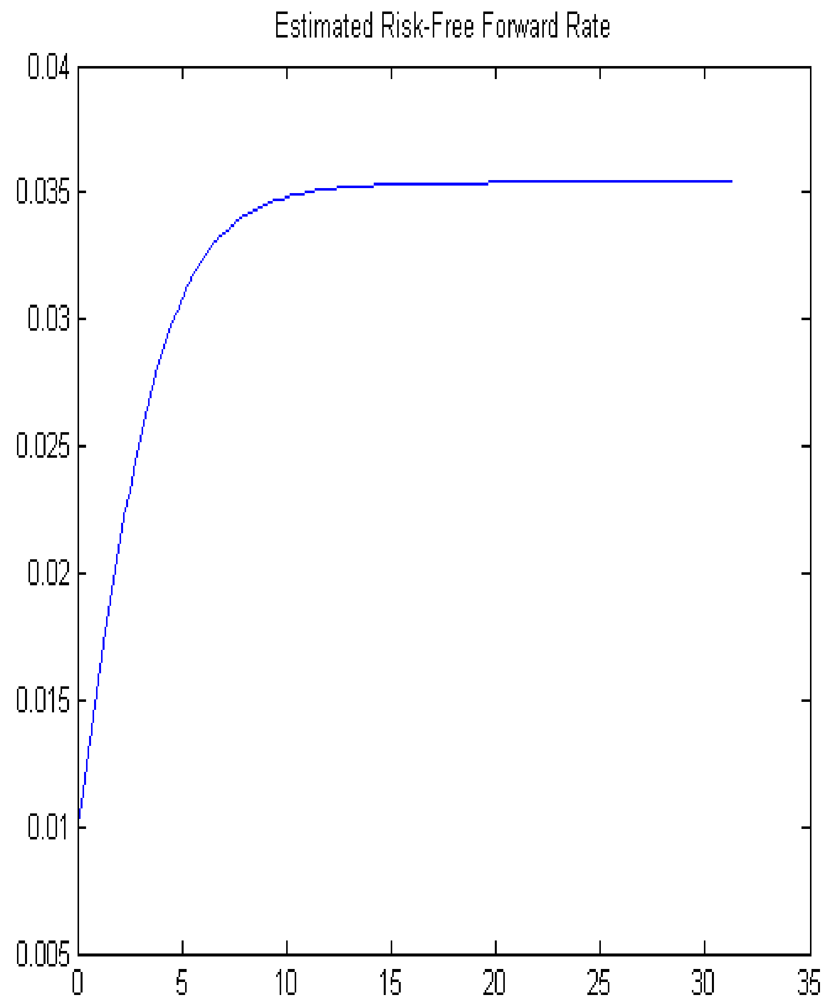
Risk-free zero-coupon yield



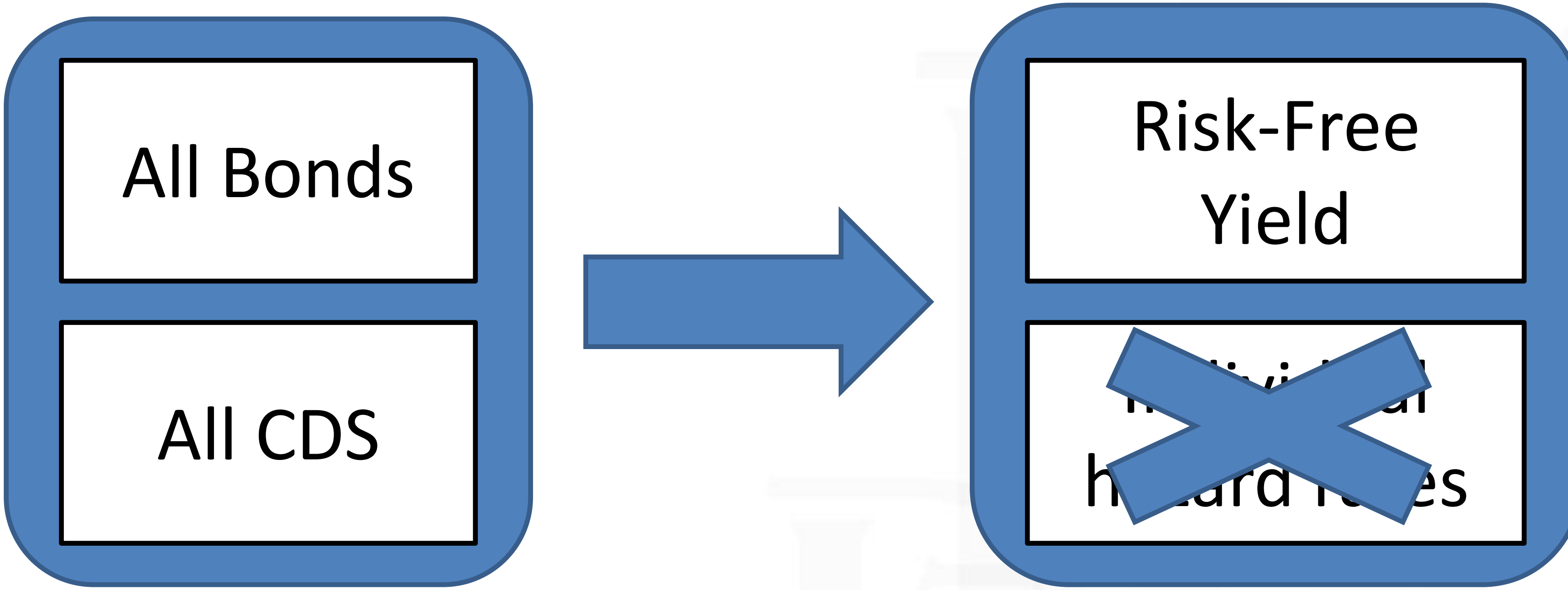


Risk-free zero-coupon yield: CIR

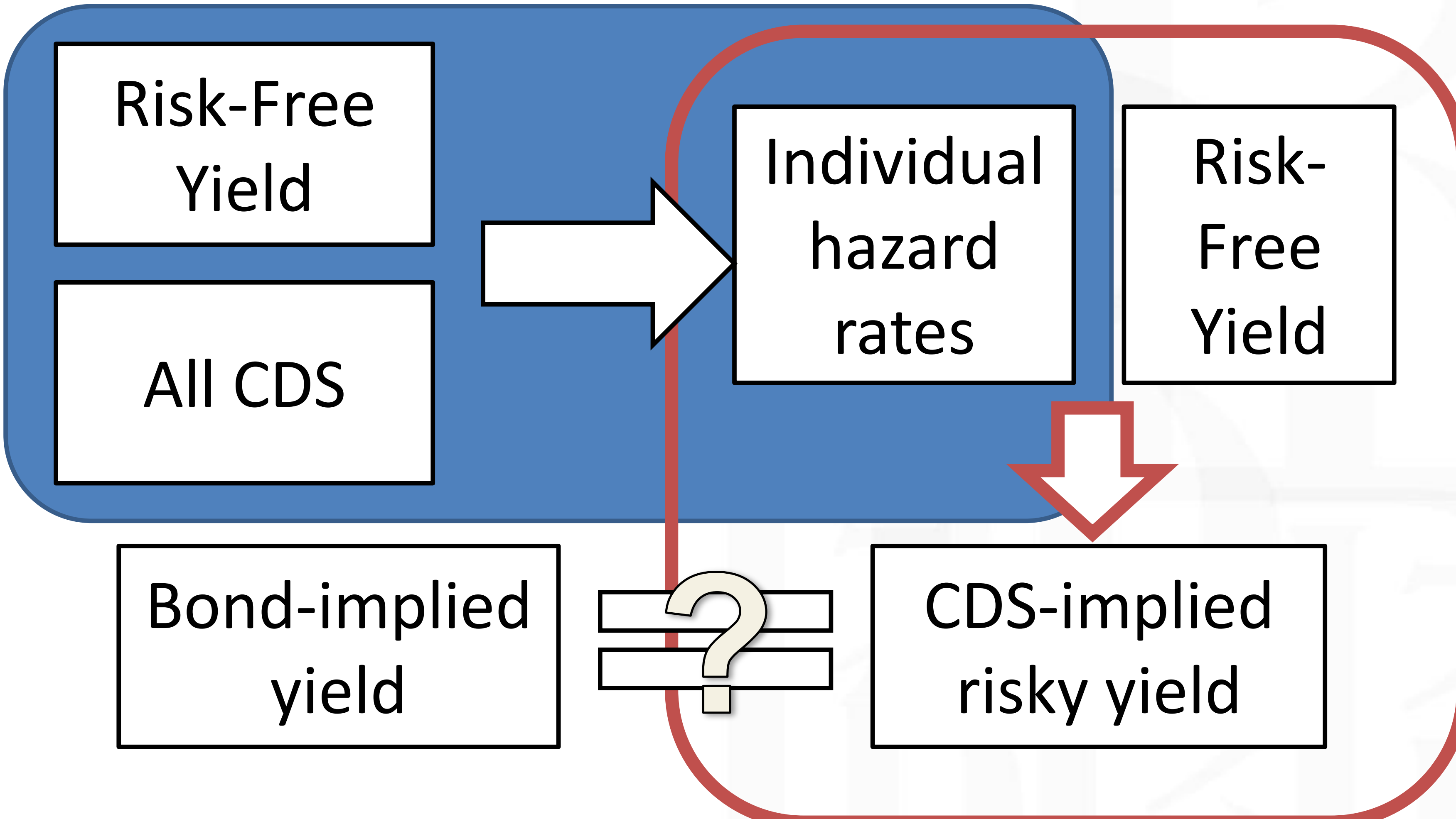




Risk-Free Yield



CDS-Implied vs Bond-Implied Yields



Discussion

- While pricing less risky bonds adequately, CDS are overly pessimistic about distressed issuers.
- Or is the bond market overly optimistic? ☺

- Without CDS bid/ask quotes it works only one way.

CIR vs Non-Parametric

- The two approaches really yield the same snapshot results, especially for hazard rates.
- This may be because CIR is in fact used to price CDS (spot hazard rates are fit almost exactly, with usual **relative** errors of several b.p.)
- Still, non-parametric model offers dynamic advantages.

Multiple factors are the future

- In a recent observation van Deventer (Kamakura Co.) noted that the realized interest rate curves have been consistent with a single-factor interest rate model only on 25% of trading days since 1962.
- Forward rate movements are consistent only on 5% of trading days.
- http://www.kamakuraco.com/Blog/tabid/231/EntryId/350/Pitfalls-in-Asset-and-Liability-Management-One-Factor-Term-Structure-Models-and-the-Libor-Swap-Curve.aspx

Thank you
for your attention!

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